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and income: a proof**

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# The Cutoff Policy of Taxation When CRRA Taxpayers Differ in Risk Aversion Coefficients and Income: a Proof

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## Abstract

Under a cutoff policy, taxpayers can either report income as usual and run the risk of being audited, or report a “cutoff” income and hence pay a threshold tax that guarantees not being audited. Whereas the mainstream literature in this field assumes risk neutrality of taxpayers – with some notable exceptions like Chu (1990) and Glen Ueng and Yang (2001) – this paper assumes risk aversion instead: taxpayers have a Constant Relative Risk Aversion (CRRA) utility function and differ in terms of their relative risk aversion coefficient and income. The novel contribution of this work is that, under certain conditions, the cutoff is accepted by taxpayers with intermediate characteristics in terms of income and relative risk aversion. Contrary to the standard result in the literature, a full separation of types (the rich who accept the cutoff versus the poor who refuse it) does not arise. However, our results confirm that the cutoff policy violates equity, as only some taxpayers directly benefit. Nonetheless, the perception of this drawback may in practice be obfuscated because that exclusion does not necessarily affect only the poor.

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**Key words:** cutoff, tax evasion, relative risk aversion.

## 1 Introduction

Under a cutoff policy, the Tax Administration audits, with a given probability, each taxpayer reporting income below a given threshold; no audit takes place, instead, of taxpayers whose income report meets the threshold. If taxpayers are risk-neutral, and the expected sanction for evasion large enough, the effect of the cutoff rule is that taxpayers whose income (and thus whose tax<sup>1</sup>) is lower than the threshold pay their tax due, thereby risking audits, while those who owe tax equal to or higher than the threshold pay the threshold tax and avoid audits. The overall payments made by taxpayers are non decreasing in income. Many different aspects of this approach have been examined in the literature (see, *e.g.*, Reinganum and Wilde, 1985, Scotchmer, 1987, Cremer, Marchand and Pestieau, 1990, Sanchez and Sobel 1993, and, for a generalization, Chander and Wilde, 1998), where it is by and large considered as an efficient strategy in agency models in which the Tax Administration as a principal can commit to a given audit policy. The cutoff rule entails efficiency gains as long as it secures savings in terms of audit costs that exceed the revenue losses in taxation. It has, however, been criticized from an equity point of view, because it introduces a regressive bias, as taxpayers’ payments strictly increase in income only until the threshold level.

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<sup>1</sup>For the sake of simplicity the income tax is described as a function of reported income, disregarding possible differences between reported (gross) income and net taxable income due to exemptions, deductions etc.

While the aforementioned literature focusses upon the efficient design of the whole tax system, including the choice of tax rates, penalties etc., Chu (1990) studies the cutoff policy as a reform that can be applied to actual tax systems. He assumes that taxpayers are risk averse, while the tax rate and the enforcement parameters are given and are such that the expected yield of tax evasion is positive. Chu shows that the introduction of a scheme implementing the cutoff policy<sup>2</sup> gives rise to a Pareto improvement. In fact, when taxpayers evade and are risk averse, the cutoff system can play a role which is impossible under risk neutrality: that of collecting risk premia for the insurance against audits provided by the cutoff. This characteristic may render the cutoff policy profitable even disregarding the benefits in terms of reducing the number of (costly) audits to be run. On the other hand, the cutoff policy still implies a regressive bias.

The Chu model has been generalized by Glen Ueng and Yang (2001). They show that the lump sum nature of the threshold tax implies that the cutoff is also efficiency improving when income depends on the labor-leisure choice of the agent. Moreover their approach allows for heterogeneous preferences. The authors, however, do not investigate thoroughly along this direction, thus failing to ascertain some aspects specifically related to heterogeneity of preferences. In their brief mention to the heterogeneous framework (Remark 1, p. 88) it is implicitly assumed that the highest income taxpayer contributes the highest expected tax revenue (including sanctions) in a standard tax system. This feature, that they show to hold under homogeneous preferences, is implicitly maintained also when preferences are heterogeneous (see also note 9, p. 93). In the latter case, however, the expected tax revenue depends on both the taxpayer's income and on her preferences, and thus a monotonic relationship between expected tax revenue and income might not hold any more. If this possibility is taken into account, the design of a Pareto improving cutoff policy becomes more demanding in terms of information: it is not enough to rely upon data pertaining to the income distribution. Preferences must be considered as well. Moreover, in this more general model, the consequences of the cutoff policy in terms of equity also deserve a closer examination.

In this paper, by building upon the models of Chu (1990) and of Glen Ueng and Yang (2001), we aim at studying the cutoff policy by explicitly taking into consideration heterogeneity of preferences. We assume that taxpayers differ in relative risk-aversion coefficient and in income, which are treated as exogenous continuous variables. To overcome the technical difficulties that ensue, the taxpayers' utility function is assumed to be CRRA (Constant Relative Risk-Aversion). We believe, however, that this loss of generality with respect to the Glen Ueng and Yang model, which refers to the whole family of utilities exhibiting risk-aversion, is adequately compensated by our main finding (Proposition 1), which heavily exploits the advantage of having a measure of heterogeneity of preferences expressed by means of relative risk-aversion coefficients. Moreover, such parametrization allows for the representation of a continuum of taxpayers, thus enabling us from this point of view to follow a more general approach<sup>3</sup> with respect to Glen Ueng and Yang, who assume a finite set of taxpayers.

The main new finding of this work pertains to the reaction of taxpayers who are requested to make a cutoff payment larger than the tax they would pay under the standard tax rule. We show that acceptance of the threshold tax among this group might leave aside two tails: the poorest with high risk-aversion and the richest with low risk-aversion. Hence the cutoff policy introduces a trade-off between efficiency advantages (Government revenue increases thanks to taxpayers' voluntary payments) and negative effects in terms of equity. The latter, however, differ from those pointed out in previous models, which predict that the cutoff policy is accepted by (all) the richest.

The paper is organized as follows. In Section 2 we describe the taxpayer's problem with reference

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<sup>2</sup>The so called FATOTA scheme provides that taxpayers can either pay a fixed amount of taxes (FAT), being exempted from tax audits, or pay taxes as usual, running the risk of tax audits (TA).

<sup>3</sup>The continuum representation is mainly followed in the literature; see, *e.g.*, Chander and Wilde (1998), who shortly review the previous studies.

to both the optimal income report and the conditions needed for acceptance of a cutoff proposal. In Section 3 we characterize those who pay the threshold tax, in terms of both true income and relative risk-aversion. Section 4 contains two examples that illustrate the main result of Section 3. Section 5 discusses the relevance of our result: while the drawbacks in terms of equity introduced by cutoff programs already known in the literature are confirmed, our contribution provides a more detailed and complex scenario in which emerges that equity is not being affected, as widely assumed, monotonically; specifically, when taxpayers are heterogeneous and have CRRA preferences some rich taxpayers may not benefit from the cutoff. Finally, the whole Section 6 is devoted to the technical proof of our main result.

## 2 The Taxpayer's Problem

Consider an economy in which there is a continuum of taxpayers. The utility that each taxpayer enjoys out of her exogenously given and non observable income  $w$  is assumed to be of the standard CRRA form, with constant relative risk-aversion coefficient  $\alpha$ :

$$u(w) = \frac{w^{1-\alpha} - 1}{1 - \alpha}. \quad (1)$$

In class (1) we also include the case  $\alpha = 1$  by taking  $u(w) = \lim_{\alpha \rightarrow 1} (w^{1-\alpha} - 1) / (1 - \alpha) = \ln w$ . Hence, the taxpayers populating our economy are indexed by their relative risk-aversion coefficient  $\alpha > 0$ .

### 2.1 Facing the Standard Tax Rule

Let us examine the taxpayer's optimal report, making reference only to general tax rules and setting aside the cutoff policy for the moment. A proportional tax system is considered: the income tax is given by  $t(y) = ty$ , where  $y$  denotes the *reported income* and  $0 < t < 1$ . We also assume that the sanction to be paid in case of audit<sup>4</sup> is proportional to the amount of the evaded tax:

$$S(w, y) = (1 + s)t(w - y), \quad (2)$$

where  $s > 0$  is a penalty rate.

As we rule out rewards to honest taxpayers by assumption,<sup>5</sup> a taxpayer will report  $y \leq w$ , where  $w > 0$  denotes the *true income*. A rational taxpayer who earned a true income  $w$  will choose to report the income  $y^*$  that maximizes her expected utility

$$\mathbb{E}u(y) = \frac{(1 - p)(w - ty)^{1-\alpha} + p[w - ty - (1 + s)t(w - y)]^{1-\alpha} - 1}{1 - \alpha} \quad (3)$$

with respect to  $y$ , where  $0 < p < 1$  is the probability of detection. Note that, by considering  $u(w) = \ln w$  when  $\alpha = 1$ ,  $\mathbb{E}u(y)$  is well defined for all  $\alpha > 0$  and for all feasible  $y$ .

The feasible set contains values for  $y$  such that  $(1 + s)t(w - y) < w - ty$ ; that is, we assume that the taxpayer can always bear the loss in case of detected evasion. A lower bound for the feasible

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<sup>4</sup>We maintain the standard assumption that detection of tax evasion occurs with probability 1 whenever the tax report is false and an audit is run.

<sup>5</sup>This is also a standard assumption, even if the theory of optimal auditing provides reasons in favor of rewards to audited honest risk-averse taxpayers (see Mookherjee and Png, 1989).

reported income  $y$  is thus  $m_w = [(1 + s)t - 1]w / (ts)$ . Since we are interested in a strictly positive income report,  $y > 0$ , we shall assume that

$$(1 + s)t > 1. \quad (4)$$

This implies that sanctions are large enough to exclude full evasion.<sup>6</sup> Therefore, the feasible set of values for the reported income  $y$  is the interval  $(m_w, w]$ , with  $m_w > 0$ .

In accordance with empirical evidence, we assume that the tax system parameters have values such that cheating in reporting income has a positive expected return. In other words, the expected sanction is assumed to be less than the expected gain for each dollar invested in tax evasion:

$$sp < 1 - p. \quad (5)$$

Hence, the case of full compliance,  $y = w$ , is also ruled out, as can readily be seen by noting that the limit of the marginal expected utility  $[\mathbb{E}u(y)]'$  as  $y \rightarrow w^-$  is negative whenever (5) holds.

Since, on the other hand,  $\lim_{y \rightarrow m_w^+} [\mathbb{E}u(y)]' = +\infty$  and  $\mathbb{E}u(y)$  is strictly concave over  $(m_w, w)$  for all  $\alpha > 0$ , there exists a unique (interior) value  $y^*$ ,  $m_w < y^* < w$ , that maximizes the expected utility, which is completely characterized in terms of F.O.C. applied to (3):

$$\frac{w - ty^* - (1 + s)t(w - y^*)}{w - ty^*} = \left( \frac{ps}{1 - p} \right)^{\frac{1}{\alpha}}. \quad (6)$$

By solving (6) for  $y^*$ , the optimal reported income proves to be a fixed share of the taxpayer's true income  $w$ , which depends on the risk-aversion coefficient  $\alpha$ :

$$y^* = \frac{(1 + s)t + \left( \frac{ps}{1 - p} \right)^{\frac{1}{\alpha}} - 1}{t \left[ s + \left( \frac{ps}{1 - p} \right)^{\frac{1}{\alpha}} \right]} w. \quad (7)$$

Note that a higher risk-aversion implies a larger share. Unlike Glen Ueng and Yang (2001) approach, in our setting a richer taxpayer (endowed with a larger  $w$ ), might have a lower reported income if her relative risk-aversion coefficient  $\alpha$  is lower. The F.O.C. condition in (7) allows for a substantial refinement of the representation of the population of taxpayers in the economy by adding a dimension to their relative risk-aversion coefficient index  $\alpha$ .

An inverse formulation of (7), which will be exploited later on, gives a relation expressing the true income of (optimizing) taxpayers, which is private information, as a function of their relative risk-aversion coefficient  $\alpha$  for any given optimal report  $y^*$ :

$$w(y^*, \alpha) = \frac{\left[ s + \left( \frac{ps}{1 - p} \right)^{\frac{1}{\alpha}} \right] ty^*}{\left( \frac{ps}{1 - p} \right)^{\frac{1}{\alpha}} + (1 + s)t - 1}. \quad (8)$$

As can easily be checked, the function  $w(y^*, \alpha)$  in (8) is strictly decreasing with respect to  $\alpha$ .

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<sup>6</sup>The literature that considers optimal income reporting under risk aversion has routinely focussed upon strictly positive reports, see, *e.g.*, Allingham and Sandmo (1972). For the role of the assumption  $(1 + s)t > 1$  in order to ensure an internal solution, also see the Appendix in Chu (1990).

## 2.2 Introducing a Cutoff

Now, let us assume that the Tax Administration offers the possibility of paying the cutoff amount  $c$  in order to avoid audits with certainty. To facilitate the description of the reaction to this proposal by a taxpayer who would pay an amount  $ty^*$  under general rules, let us break down  $c$  as follows:

$$c = ty^* + x. \quad (9)$$

Whenever  $x$  is negative, that is the cutoff is lower than the ordinary tax, the trivial implication is that the taxpayer chooses the cutoff.<sup>7</sup> On the contrary, if the taxpayer cannot afford paying  $x$ , that is:

$$x \geq w - ty^*$$

she always prefers ordinary taxation.

We will focus upon the most interesting case:  $x$  is positive and the taxpayer can afford it. In this case, the taxpayer will accept the offer if she is at least indifferent as to whether to pay the requested amount  $c$  or to pay only  $ty^*$  and risk an audit. Thus, for a given  $x > 0$ , she chooses to pay the threshold tax if

$$\frac{[w - (ty^* + x)]^{1-\alpha}}{1-\alpha} \geq \frac{(1-p)(w - ty^*)^{1-\alpha} + p[w - ty^* - (1+s)t(w - y^*)]^{1-\alpha}}{1-\alpha}, \quad (10)$$

where the additive constants  $-(1-\alpha)^{-1}$  have already been dropped from both sides.

By jointly considering the optimal condition (8) and the threshold condition (10), we are led to the following system:

$$\begin{cases} w = \left\{ \left[ s + \left( \frac{ps}{1-p} \right)^{\frac{1}{\alpha}} \right] ty^* \right\} / \left[ \left( \frac{ps}{1-p} \right)^{\frac{1}{\alpha}} + (1+s)t - 1 \right] \\ \frac{(w - ty^* - x)^{1-\alpha}}{1-\alpha} \geq \frac{(1-p)(w - ty^*)^{1-\alpha} + p[w - ty^* - (1+s)t(w - y^*)]^{1-\alpha}}{1-\alpha} \end{cases} \quad (11)$$

The first equation of system (11) links the taxpayer's true income  $w$  to  $y^*$  according to (8) – or, equivalently, according to (6). The second equation is the weak preference condition for paying the threshold tax instead of reporting  $y^*$  and risking an audit. All pairs  $(\alpha, w)$  solving system (11) characterize in terms of relative risk-aversion  $\alpha$  and true income  $w$  the subset of taxpayers whose optimal report is  $y^*$  and who prefer to pay the threshold tax  $c$  although it is larger than the tax they would pay under general rules.

In the following sections we will take the optimal report  $y^*$  and the ‘premium’  $x$  as given and  $\alpha$  and  $w$  as the unknowns and provide a characterization of the set of taxpayers accepting the cutoff.

## 3 The Main Result

By plugging the first equality in system (11) into the second inequality, after some tedious algebra we obtain a single inequality where the unknown is the sole variable  $\alpha$ :

$$x \leq \frac{(1+s)(1-t)ty^*}{\left( \frac{ps}{1-p} \right)^{\frac{1}{\alpha}} + (1+s)t - 1} \left\{ 1 - \left[ 1 - p + p \left( \frac{ps}{1-p} \right)^{\frac{1-\alpha}{\alpha}} \right]^{\frac{1}{1-\alpha}} \right\}. \quad (12)$$

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<sup>7</sup>The cutoff is always preferred as long as it is lower than  $ty^* + p(1+s)t(w - y^*)$ , *i.e.*, the expected payment in terms of taxes and fines. While writing  $c = ty^* + p(1+s)t(w - y^*) + x$  would be preferable in order to highlight the possible request of a risk premium, we choose a simpler breaking up of term  $c$  in (9) to ease algebraic reduction.

Inequality (12) characterizes all taxpayers with relative risk-aversion index  $\alpha$  who (optimally) reported income  $y^*$  and choose to participate in the cutoff program for an extra amount  $x$  when  $0 < t < 1$  is the tax rate,  $s > 0$  is the penalty rate and  $0 < p < 1$  is the probability of detection under general rules.

For future algebraic convenience we introduce the following changes of parameters:

$$\beta = \frac{x}{(1-t)y^*}; \quad (13)$$

$$\gamma = \ln \left( \frac{ps}{1-p} \right); \quad (14)$$

$$\delta = \frac{1}{(1+s)t} = \frac{p}{t[p + (1-p)e^\gamma]}. \quad (15)$$

Parameter  $\beta$  transforms component  $x$  in percentage terms with respect to the reported income after taxation under general rules, we shall assume  $\beta \leq 1$ ;  $\gamma$  is a transformation of the sanction rate  $s$  which takes into account also the probability of detection  $p$  ( $\gamma$  and  $p$  will be the key parameters in our analysis); finally,  $\delta$  transforms the tax rate  $t$  and the sanction rate  $s$  and, by using (14), can be written as a function of  $\gamma$  and the probability of detection  $p$ , a form that will be exploited in Section 6.

Consider the function of the sole variable  $\alpha$  defined by

$$\tau(\alpha) = \frac{1 - \exp \left[ \frac{\ln \left( 1 - p + pe^{\gamma \frac{1-\alpha}{\alpha}} \right)}{1 - \alpha} \right]}{1 - \delta \left( 1 - e^{\frac{\gamma}{\alpha}} \right)}. \quad (16)$$

Since, by l'Hôpital's rule,  $\lim_{\alpha \rightarrow 1} \left[ \ln \left( 1 - p + pe^{\gamma \frac{1-\alpha}{\alpha}} \right) / (1 - \alpha) \right] = p\gamma$ , when  $\alpha = 1$  (16) boils down to

$$\tau(1) = \frac{1 - e^{p\gamma}}{1 - \delta(1 - e^\gamma)}.$$

In order to let  $\tau$  be defined for all  $\alpha > 0$  we shall take into account the discussion on parameters  $x$ ,  $s$ ,  $p$  and  $t$  developed in the previous section, which translates into the following conditions which will hold throughout the paper:  $0 < \beta \leq 1$ ,  $\gamma < 0$  and  $0 < \delta < 1$ . Specifically,  $\gamma < 0$  follows from (5) and  $\delta < 1$  is a consequence of (4).

The number  $\tau(\alpha)$  can be interpreted as the individual (percent) threshold value for the taxpayer characterized by coefficient of risk aversion  $\alpha$ : if  $\beta$  is smaller or equal to  $\tau(\alpha)$  such taxpayer opts for the cutoff. Inequality (12) is thus equivalent to the following:

$$\tau(\alpha) \geq \beta, \quad (17)$$

whose solution set contains all taxpayers characterized by relative risk aversion coefficient  $\alpha$  who chooses (or are indifferent to) to pay the cutoff rather than incurring the risk of being audited when the (percent) cutoff offered by the tax administration is  $\beta$ . Note that parameter  $\beta$  embeds the optimal report in absence of cutoff,  $y^*$ , which, in turn, depends on the true income  $w$ ; therefore, the only relevant independent variable left is the relative risk aversion coefficient  $\alpha$  representing heterogeneity in the population of taxpayers.

For each given value  $0 < \beta \leq 1$ , inequality (17) defines the *upper contour set* of  $\beta$  for the function  $\tau$ ; our goal is to show that, under some conditions on the parameters, such upper contour sets are intervals not necessarily having zero as their left endpoint. While we will be able to prove that  $\tau$  is quasiconcave in some cases [when values of  $\beta$  large enough,  $\beta > \sup \{\tau(\alpha) : \alpha > 0\}$ , are allowed], in the general case we will establish the desired property of the upper contour sets only for values of  $\beta$  not too large [ $\beta < \sup \{\tau(\alpha) : \alpha > 0\}$ ].

The following technical assumption further restricts the admissible values of the parameters.

**A.1** Parameters  $\beta$  and  $\gamma$  have values in the following ranges:  $0 < \beta \leq 1$  and<sup>8</sup>  $6/(\sqrt{3}-3) \leq \gamma \leq -2$ . Moreover, the probability of detection  $p$  and the tax rate  $t$  must satisfy

$$\delta = \frac{p}{t[p + (1-p)e^\gamma]} \leq 1 + \frac{\gamma e^\gamma}{(1-e^\gamma)^2}. \quad (18)$$

Note that the RHS in (18) is less than 1 as  $\gamma$  is negative. By expliciting  $p$ , (18) assumes the following more cumbersome form which separates parameter  $p$  from  $t$  and  $\gamma$ :<sup>9</sup>

$$p \leq \frac{\left[1 + \frac{\gamma e^\gamma}{(1-e^\gamma)^2}\right] e^\gamma t}{1 - \left[1 + \frac{\gamma e^\gamma}{(1-e^\gamma)^2}\right] (1-e^\gamma) t}. \quad (19)$$

**Proposition 1** Suppose Assumption A.1 holds true.

i) If  $\gamma = -2$  and

$$\beta \leq \frac{1 - \exp(-\sqrt{3.9781p})}{1 - \delta}, \quad (20)$$

then the upper contour sets of  $\tau$  defined in (16) are intervals. For  $\beta$  large enough, such intervals are disjoint from the origin, i.e., their left endpoint is strictly positive.

ii) If  $6/(\sqrt{3}-3) \leq \gamma < -2$ , then, for each  $\beta$  satisfying

$$\beta \leq \frac{1 - \exp\left\{\frac{2 \ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}}{1 - (1 - e^{-2})\delta}, \quad (21)$$

the solution set of inequality (17) is a nonempty interval. Again, for  $\beta$  large enough, such interval is disjoint from the origin, i.e., its left endpoint is strictly positive.

Section 6 is entirely devoted to the proof of Proposition 1.

Note that (21) is more restrictive than (20); specifically, as it will be seen in the proof, the RHS in (21) is always less than  $\sup\{\tau(\alpha) : \alpha > 0\}$ . On one side this is sufficient for nonemptiness of the solution of (17), but is not enough to establish that  $\tau$  is quasiconcave.

## 4 Two Numerical Examples

Let us apply Proposition 1 for the following values of parameters:  $\gamma = -4.0644$ ,  $p = 0.01$ , to which corresponds a sanction rate  $s = (1-p)e^\gamma/p \simeq 1.7$ , and  $t = 0.44$ . Assumption A.1 is satisfied as  $\delta = p/\{t[p + (1-p)e^\gamma]\} \simeq 0.8417 < 0.9277 \simeq 1 + \gamma e^\gamma/(1-e^\gamma)^2$ . Since  $\gamma < -2$ , part (ii) of Proposition 1 is involved and we have only to establish the upper bound for the (percent) cutoff value given by condition (21):  $\beta \leq \bar{\beta} = \left\{1 - e^{\frac{2 \ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}}\right\} / [1 - (1 - e^{-2})\delta] \simeq 0.1434$ ; for example, by (13),  $\bar{\beta}$  corresponds to a value  $x = (1-t)y^*\bar{\beta} \simeq 80.299$  when  $y^* = 1000$ . Therefore, any fixed (percent) premium that satisfies  $\beta \leq 0.1434$  produces a nonempty interval – either of the form  $(0, \alpha_r]$

<sup>8</sup>Note that  $6/(\sqrt{3}-3) \simeq -4.7321$ .

<sup>9</sup>As prescribed by (14), parameter  $\gamma$  actually contains parameter  $p$  in its expression; however, rather than being a function of  $p$ ,  $\gamma$  must be interpreted as a function of the original sanction rate  $s$  for any given value of  $p$ , and thus as a parameter which is independent of parameter  $p$  itself. In this perspective, we can say that  $p$  and  $\gamma$  are “separated” in the RHS of (19).



or of the form  $[\alpha_\ell, \alpha_r]$  with  $\alpha_\ell > 0$  – of relative risk-aversion coefficients characterizing agents who pay the threshold tax. For example, with  $\beta = 0.13$  (corresponding to  $x = 72.8$  when  $y^* = 1000$ ), the interval has  $\alpha_\ell \simeq 0.53 > 0$  and  $\alpha_r \simeq 3.93$  as endpoints, as shown in figure 1 where the function  $\tau$  defined in (16) is plotted for our values for the parameters. If  $y^* = 1000$  these two values, correspond to a minimum true income<sup>10</sup>  $w \simeq 1664$  (corresponding to  $\alpha_r \simeq 3.93$ ) and a maximum true income  $w \simeq 3969$  (corresponding to  $\alpha_\ell \simeq 0.53$ ), which imply evasion (in terms of share of concealed income when  $y^*$  is reported) of around 40% and around 75% respectively.

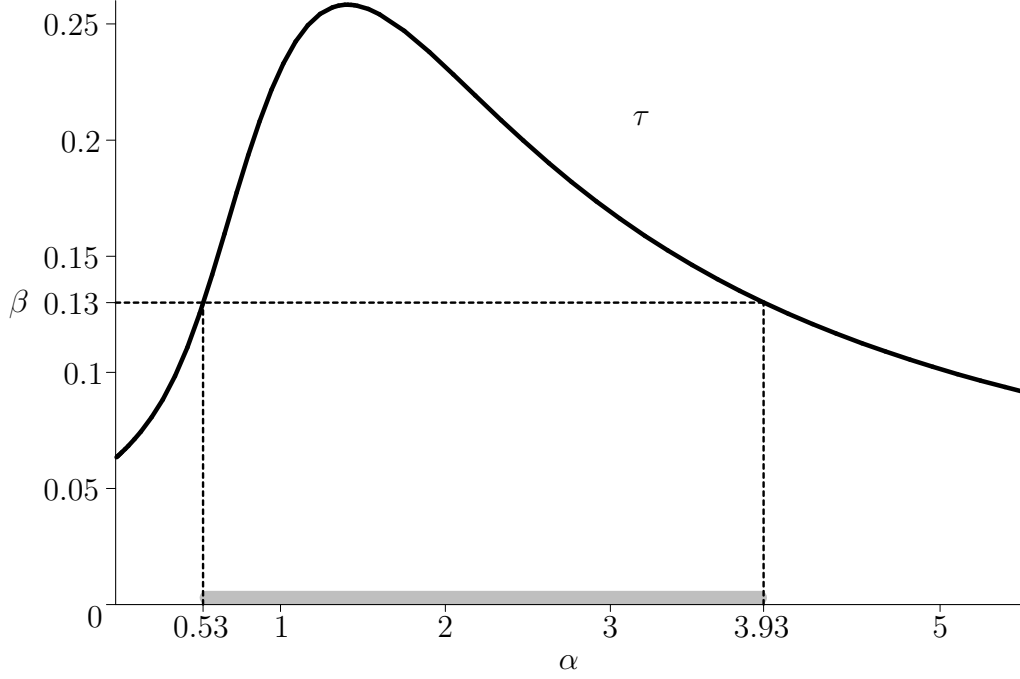


FIGURE 1: the solution set of inequality (17) for  $\gamma = -4.0644$ ,  $p = 0.01$ ,  $s = 1.7$ ,  $t = 0.44$  and  $\beta = 0.13$ .

Figure 1 shows that more than the statement of part (ii) of Proposition 1 is true; the shape of function  $\tau$  is striking: it is clearly a quasiconcave function, that is, its upper level sets are intervals for all  $\beta \leq \max \{\tau(\alpha) : \alpha > 0\} \simeq 0.26$ , not only for  $\beta \leq 0.1434$ . However, we have not been able to establish this property in general, at least for  $\gamma < -2$ . Only when  $\gamma = -2$  and part (i) of Proposition 1 applies, and thus  $\beta$  turns out to be constrained by the much looser condition (20) rather than by condition (21), it can be established that  $\tau$  is actually quasi concave for some cases, as it may happen that  $\max \{\tau(\alpha) : \alpha > 0\}$  is smaller than the RHS in (20).

For example, with  $\gamma = -2$ ,  $p = 0.05$ , to which corresponds a sanction rate  $s = (1 - p)e^{-2}/p \simeq 2.57$ , and  $t = 0.44$ , Assumption A.1 is satisfied as  $\delta = p / \{t[p + (1 - p)e^{-2}]\} \simeq 0.6364 < 0.638 \simeq 1 - 2e^{-2} / (1 - e^{-2})^2$ . Part (i) of Proposition 1 applies and the upper bound for the (percent) cutoff value given by condition (20) is  $\beta \leq \bar{\beta} = [1 - \exp(-\sqrt{3.9781p})] / (1 - \delta) \simeq 0.9895$ . Figure 2 shows that  $\max \{\tau(\alpha) : \alpha > 0\} \simeq 0.23$ , well below  $\bar{\beta} \simeq 0.9895$ ; in this case part (i) of Proposition 1 can be restated by saying that the function  $\tau$  is quasiconcave for these values of parameters.

<sup>10</sup>Recall that the true income of (optimizing) taxpayers as a function of their relative risk aversion  $\alpha$  is given by (8).

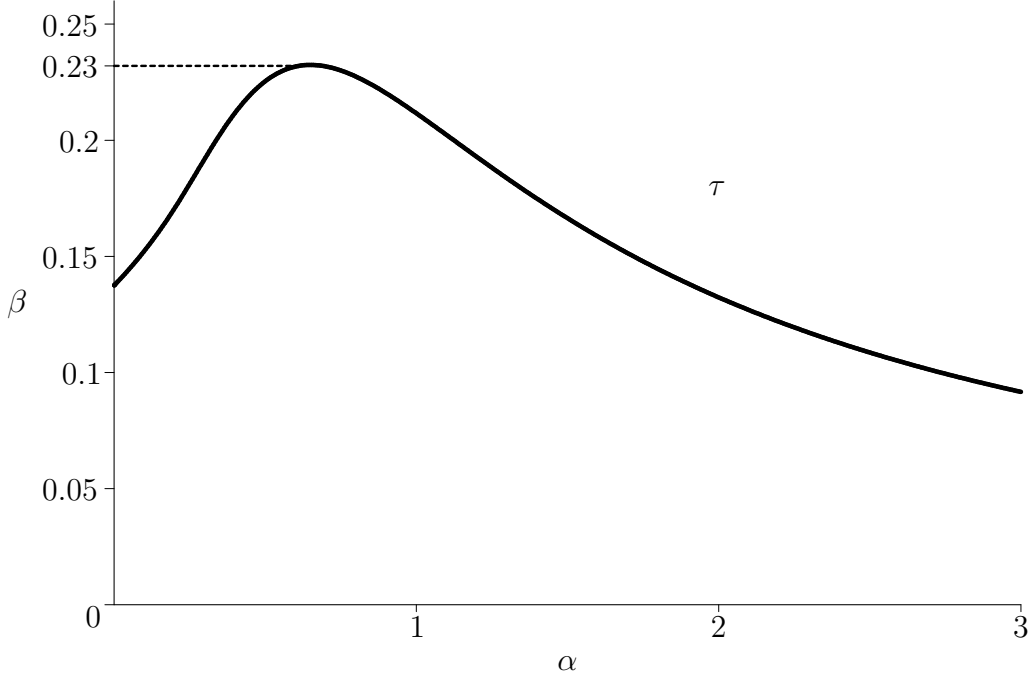


FIGURE 2: plot of the function  $\tau$  for  $\gamma = -2$ ,  $p = 0.05$ ,  $s = 2.57$  and  $t = 0.44$ .

## 5 Equity Considerations

If the solution set of inequality (17) is of the form  $(0, \alpha_r]$  – like, for instance, in the first example of Section 4 whenever  $\beta \lesssim 0.06$ , as can be understood from Figure 1 – relatively rich taxpayers with low risk-aversion, *i.e.*, with  $\alpha \leq \alpha_r$ , pay the threshold tax, while relatively poor ones with high risk-aversion, that is with  $\alpha > \alpha_r$ , do not accept the cutoff. The latter prefer to submit their optimal report  $y^*$ , which, by (7), conceals only a relatively small income amount, thereby risking audits. This result can be explained as follows. As reported income  $y^*$  approaches true income  $w$  for very risk averse taxpayers, the expected sanction decreases, while the risk premium under a CRRA utility function does not increase enough to counter the former effect.

If the solution set of inequality (17) is of the form  $[\alpha_\ell, \alpha_r]$  with  $\alpha_\ell > 0$  – like in the first example of Section 4 with  $\beta = 0.13$  – in addition to the reaction just discussed, the cutoff is also refused by taxpayers with a risk-aversion coefficient below some lower bound  $\alpha_\ell$ , whose optimal income report  $y^*$ , again by (7), conceals a relatively large income amount. These taxpayers, too, prefer risking audits rather than paying the threshold tax. In this case there are two groups of taxpayers who refuse the cutoff when it includes a premium, one characterized by relatively high true income and the other by relatively low true income. It is clear from both figures 1 and 2 that this last situation may happen only if the premium component  $\beta$  is large enough, while, at the same time, sufficiently small to allow for participation in the cutoff proposal, as part (ii) of Proposition 1 guarantees nonemptiness whenever condition (21) is satisfied.

In other words, we conclude that, as intuition suggests, the set of participants in the cutoff program shrinks as the premium asked by the tax administration to buy an insurance against the possibility of being audited increases. However, the novel contribution of the present work is that, contrary to conjectures hitherto formulated by the mainstream literature, such ‘shrinking’ does not follow a simple monotonic pattern. Specifically, we have shown that, when taxpayers have CRRA preferences, above some threshold value for  $\beta$  not only less and less poor taxpayers (identified by the decreasing

right endpoint  $\alpha_r$  of relative risk aversion index which bounds the set of participants from above) choose the cutoff because of its growing price, but also some (very) rich taxpayers characterized by low risk aversion coefficients,<sup>11</sup>  $0 < \alpha < \alpha_\ell$ , start refusing the offer as well.

Therefore, our result confirms the widely accepted criticism toward cutoff programs: equity is adversely affected. Nonetheless, such drawback assume a more multi-faceted pattern when taxpayers are risk averse, heterogeneous and endowed with a CRRA utility function. We showed that, whenever the cost of entering the cutoff is high enough, the well known regressive bias is mitigated by the refusal of the cutoff by two tails of taxpayers, the relatively poor and the relatively rich, who do not receive suitable insurance offers.

## 6 Proof of Proposition 1

The proof of Proposition 1 will be accomplished through several steps. First of all we restate inequality (17) – that is, (12) – in a more convenient form. By using (16) and rearranging terms in (17) we get

$$\frac{\ln \left( 1 - p + pe^{\gamma \frac{1-\alpha}{\alpha}} \right)}{1 - \alpha} \leq \ln \left\{ 1 - \beta \left[ 1 - \delta \left( 1 - e^{\frac{\gamma}{\alpha}} \right) \right] \right\},$$

which is equivalent to the following system:

$$\begin{cases} \ln \left( 1 - p + pe^{\gamma \frac{1-\alpha}{\alpha}} \right) \leq (1 - \alpha) \ln \left[ 1 - (1 - \delta) \beta - \delta \beta e^{\frac{\gamma}{\alpha}} \right] & \text{if } 0 < \alpha < 1 \\ \ln \left( 1 - p + pe^{\gamma \frac{1-\alpha}{\alpha}} \right) \geq (1 - \alpha) \ln \left[ 1 - (1 - \delta) \beta - \delta \beta e^{\frac{\gamma}{\alpha}} \right] & \text{if } \alpha > 1. \end{cases} \quad (22)$$

Define  $f : \mathbb{R}_{++} \rightarrow \mathbb{R}$  by

$$f(\alpha) = \phi(\alpha) + \psi(\alpha) - (1 - p), \quad (23)$$

where<sup>12</sup>

$$\phi(\alpha) = \exp \left\{ (1 - \alpha) \ln \left[ 1 - (1 - \delta) \beta - \delta \beta e^{\frac{\gamma}{\alpha}} \right] \right\}, \quad (24)$$

$$\psi(\alpha) = -pe^{\gamma \frac{1-\alpha}{\alpha}}. \quad (25)$$

Then system (22) – and thus inequalities (12) and (17) – can be written as

$$\begin{cases} f(\alpha) \geq 0 & \text{if } 0 < \alpha < 1 \\ f(\alpha) \leq 0 & \text{if } \alpha > 1. \end{cases} \quad (26)$$

The function  $f$  defined in (23) is a smooth function defined for all  $\alpha > 0$ . Note that  $f$  equals zero in  $\alpha = 1$  (corresponding to logarithmic utility in our model) for all values of parameters  $p$ ,  $\beta$ ,  $\gamma$  and  $\delta$  satisfying Assumption A.1; as a matter of fact, consistently with the algebraic manipulations of inequality (17) required to obtain system (26), the point  $\alpha = 1$  is being explicitly excluded from (26), as it does not carry useful information on its solution set. However, as the original function  $\tau$  in (16) is continuous in  $\alpha = 1$ , this point must be taken into consideration. Since the solution set of (26) is nonempty when  $f$  crosses the horizontal axis from above at  $\alpha = 1$ , we shall include  $\alpha = 1$  in the solution set as long as  $f'(1) \leq 0$ .<sup>13</sup>

<sup>11</sup>Recall that, by (8), smaller values of  $\alpha$  correspond to larger levels of true income  $w$ .

<sup>12</sup>Since  $\gamma/\alpha < 0$  for all  $\alpha > 0$ ,  $1 - (1 - \delta) \beta - \delta \beta e^{\frac{\gamma}{\alpha}}$  turns out to be always positive and thus  $\phi(\alpha)$  in (24) is well defined for all  $\alpha > 0$ .

<sup>13</sup>More precisely, a sufficient (but not necessary) condition for a nontrivial – *i.e.*, with positive Lebesgue measure – solution set of (26) is  $f'(1) < 0$ .

In principle, the condition  $f'(1) \leq 0$  does not rule out the possibility that the solution set is a disjoint union of intervals; in the rest of this section we shall prove that this cannot be the case whenever Assumption A.1 and either condition (20) or condition (21) hold, regardless of the sign of  $f'(1)$ . This will be achieved by partitioning  $\mathbb{R}_{++}$  into a number of subintervals and by studying monotonicity and curvature properties of the two functions  $\phi$  and  $\psi$ , defined in (24) and (25) respectively, over each subinterval. First, we need some preliminary lemmas.

## 6.1 Preliminary results

We start by studying the function  $\phi$  defined in (24). In the sequel we shall often split it into the composition  $\phi(\alpha) = \exp[g(\alpha)]$ , with

$$g(\alpha) = (1 - \alpha) \ln \left( A - Be^{\frac{\gamma}{\alpha}} \right), \quad (27)$$

where constants  $A$  and  $B$  are introduced in order to ease notation and are defined by:

$$A = 1 - (1 - \delta)\beta, \quad (28)$$

$$B = \delta\beta. \quad (29)$$

Note that, under Assumption 1, the following holds:

$$0 < B < A < 1. \quad (30)$$

Moreover, also the function  $g$  will be sometimes written as the product  $g(\alpha) = (1 - \alpha)h(\alpha)$ , where

$$h(\alpha) = \ln \left( A - Be^{\frac{\gamma}{\alpha}} \right). \quad (31)$$

Note that  $h(\alpha) < 0$  for all  $\alpha > 0$ .

**Lemma 1** *Let Assumption 1 holds. Then  $g'(\alpha) > 0$  for all  $\alpha > 0$ , and thus the function  $\phi$  is strictly increasing.*

**Proof.**  $g'$  has the following expression:

$$g'(\alpha) = -h(\alpha) + (1 - \alpha)h'(\alpha), \quad (32)$$

where

$$h'(\alpha) = \frac{\gamma Be^{\frac{\gamma}{\alpha}}}{\alpha^2 (A - Be^{\frac{\gamma}{\alpha}})} \quad (33)$$

is clearly negative since  $\gamma < 0$ . As both  $h$  and  $h'$  are negative, the lemma is true whenever  $\alpha \geq 1$ . Hence, let us assume  $0 < \alpha < 1$  and look for a lower bound for  $g'$  which is independent of  $\alpha$ .

We first compute a lower bound for  $h'$ . A direct computation of  $h''$  yields:

$$\begin{aligned} h''(\alpha) &= -\frac{\gamma Be^{\frac{\gamma}{\alpha}}}{\alpha^2 (A - Be^{\frac{\gamma}{\alpha}})} \left[ \frac{2}{\alpha} + \frac{\gamma}{\alpha^2} + \frac{\gamma Be^{\frac{\gamma}{\alpha}}}{\alpha^2 (A - Be^{\frac{\gamma}{\alpha}})} \right] \\ &= -\left(2 + \frac{\gamma}{\alpha}\right) \frac{h'(\alpha)}{\alpha} - [h'(\alpha)]^2. \end{aligned}$$

Since by Assumption 1  $\gamma \leq -2$ ,  $2 + \gamma/\alpha \leq 0$  holds for all  $0 < \alpha < 1$  and thus  $h''$  turns out to be strictly negative; that is,  $h'$  is strictly decreasing and we can take  $h'(1) = (A - Be^\gamma)^{-1} \gamma Be^\gamma$ ,

which is independent of  $\alpha$ , as its lower bound. As  $1 - \alpha < 1$  and  $h'(1) < 0$ , the second term in (32) is bounded from below by  $h'(1)$ . Moreover, since  $-h$  is an increasing function of  $\alpha$  [recall that  $h' < 0$ ],  $-h(\alpha) > \lim_{\alpha \rightarrow 0^+} [-h(\alpha)] = -\ln A$  for all  $0 < \alpha < 1$ . Hence,  $g'$  is bounded from below by  $-\ln A + h'(1)$ , that is,

$$g'(\alpha) > -\ln A + \frac{\gamma B e^\gamma}{A - B e^\gamma}, \quad (34)$$

where the RHS is independent of  $\alpha$ .

The first term in the RHS of (34) is positive, while the second term is negative; thus, now we need to find conditions under which the RHS is nonnegative. Specifically, by expanding constants  $A$  and  $B$  as in (28) and (29) respectively, we must find for what values of parameters  $p, \beta, \gamma$  and  $\delta$  the following inequality holds:

$$-\ln[1 - (1 - \delta)\beta] + \frac{\gamma\delta\beta e^\gamma}{1 - (1 - \delta)\beta - \delta\beta e^\gamma} \geq 0. \quad (35)$$

Since it clearly holds (with equality) for  $\beta = 0$ , a sufficient condition is that the derivative with respect to  $\beta$  of the LHS be nonnegative for all  $0 < \beta \leq 1$ . A direct computation yields

$$\frac{\partial}{\partial \beta} \left\{ -\ln[1 - (1 - \delta)\beta] + \frac{\gamma\delta\beta e^\gamma}{1 - (1 - \delta)\beta - \delta\beta e^\gamma} \right\} = \frac{1 - \delta}{1 - (1 - \delta)\beta} + \frac{\gamma\delta e^\gamma}{[1 - (1 - \delta)\beta - \delta\beta e^\gamma]^2};$$

in order to guarantee that such expression is nonnegative, we rearrange terms and study the following inequality:

$$\frac{1 - \delta}{\gamma\delta e^\gamma} + \frac{1 - (1 - \delta)\beta}{[1 - (1 - \delta)\beta - \delta\beta e^\gamma]^2} \leq 0. \quad (36)$$

We now show that the second term in the LHS of (36) is increasing in  $\beta$ :

$$\frac{\partial}{\partial \beta} \left\{ \frac{1 - (1 - \delta)\beta}{[1 - (1 - \delta)\beta - \delta\beta e^\gamma]^2} \right\} = \frac{(1 - \delta)[1 - (1 - \delta)\beta] + [2 - (1 - \delta)\beta]\delta e^\gamma}{[1 - (1 - \delta)\beta - \delta\beta e^\gamma]^3} > 0,$$

as both the numerator and the denominator in the RHS are positive. Thus, an upper bound for the LHS in (36) is obtained by letting  $\beta = 1$ :

$$\frac{1 - \delta}{\gamma\delta e^\gamma} + \frac{1 - (1 - \delta)\beta}{[1 - (1 - \delta)\beta - \delta\beta e^\gamma]^2} \leq \frac{1 - \delta}{\gamma\delta e^\gamma} + \frac{\delta}{(\delta - \delta e^\gamma)^2} = \frac{1 - \delta}{\gamma\delta e^\gamma} + \frac{1}{\delta(1 - e^\gamma)^2}.$$

Therefore, a sufficient condition for (36) is the following:

$$\frac{1 - \delta}{\gamma\delta e^\gamma} + \frac{1}{\delta(1 - e^\gamma)^2} \leq 0 \quad \Longleftrightarrow \quad \delta \leq 1 + \frac{\gamma e^\gamma}{(1 - e^\gamma)^2},$$

which is condition (18) of Assumption 1. Since (36) is itself a sufficient condition for (35), which, through (34), establishes that  $g'(\alpha) > 0$  for  $0 < \alpha < 1$ , the proof is complete. ■

**Lemma 2** *Under Assumption 1 there is a unique value  $\hat{\alpha} > 0$  such that  $g''(\hat{\alpha}) = 0$ ,  $g''(\alpha) < 0$  for  $0 < \alpha < \hat{\alpha}$  and  $g''(\alpha) > 0$  for  $\alpha > \hat{\alpha}$ ;  $\hat{\alpha}$  is the unique number satisfying<sup>14</sup>*

$$\hat{\alpha} = -\frac{\gamma A}{(2 - \gamma)A - 2B e^{\frac{\gamma}{\hat{\alpha}}}}. \quad (37)$$

<sup>14</sup>The exact solution  $\hat{\alpha}$  of (37) involves the *Lambert W function* (see, e.g., Corless et al., 1996); specifically,  $\hat{\alpha} = -\gamma / [\text{LambertW}(-2B e^{\gamma-2}/A) + 2 - \gamma]$ .

Useful bounds for the value of  $\hat{\alpha}$ , as functions of the sole parameter  $\gamma$ , are given by:

$$\frac{1}{2} \leq \frac{\gamma}{\gamma - 2} < \hat{\alpha} < \frac{\gamma}{\gamma - 2 + 2e^\gamma} < 1. \quad (38)$$

Hence,  $g$  is strictly concave for  $0 < \alpha < \hat{\alpha}$  and it is strictly convex for  $\alpha > \hat{\alpha}$ , while  $\phi$  turns out to be strictly convex for  $\alpha > \hat{\alpha}$ , but no conclusion can be drawn on its curvature properties for  $0 < \alpha < \hat{\alpha}$ .

**Proof.** By using the notation (31),  $g''$  can be written as follows:

$$g''(\alpha) = -h'(\alpha) \left\{ 2 + (1 - \alpha) \left[ h'(\alpha) + \frac{2\alpha + \gamma}{\alpha^2} \right] \right\}. \quad (39)$$

Since  $h' < 0$ , it is enough to study the sign of the term in curly brackets, which is the same as that of the following expression, obtained by using (33) and after rearranging terms:

$$\zeta(\alpha) = 2 \left( A - Be^{\frac{\gamma}{\alpha}} \right) \alpha + (1 - \alpha) \gamma A. \quad (40)$$

It is easily seen that any root  $\hat{\alpha}$  of  $\zeta$  must satisfy (37).

To show that  $\hat{\alpha}$  is unique, firstly note that  $\lim_{\alpha \rightarrow 0^+} \zeta(\alpha) = \gamma A < 0$ ; moreover, since the first term in the RHS of (40) is positive and  $\gamma < 0$ ,  $\zeta(\alpha) > 0$  certainly holds for  $\alpha \geq 1$ . Thus, it is sufficient to establish that the function  $\zeta$  defined in (40) is strictly increasing for  $0 < \alpha < 1$ . To see this, let us differentiate it with respect to  $\alpha$ :

$$\frac{\partial}{\partial \alpha} \zeta(\alpha) = 2 \left( A - Be^{\frac{\gamma}{\alpha}} \right) - \gamma \left( A - \frac{2}{\alpha} Be^{\frac{\gamma}{\alpha}} \right);$$

as the first term on the RHS is positive and  $\gamma < 0$ , we just need to establish that

$$A - \frac{2}{\alpha} Be^{\frac{\gamma}{\alpha}} > 0 \quad \Longleftrightarrow \quad \frac{e^{\frac{\gamma}{\alpha}}}{\alpha} < \frac{A}{2B} \quad (41)$$

for  $0 < \alpha < 1$ . As  $(\partial/\partial \alpha) (e^{\frac{\gamma}{\alpha}}/\alpha) = -(e^{\frac{\gamma}{\alpha}}/\alpha^2) (1 + \gamma/\alpha)$ , the LHS of the last inequality is increasing whenever  $1 + \gamma/\alpha < 0 \Longleftrightarrow \alpha < -\gamma$ ; but this is certainly the case since  $0 < \alpha < 1$  and  $\gamma \leq -2$ . Thus, a sufficient condition for (41) is  $e^\gamma < A/(2B)$ , which definitely holds since, by (30),  $A/B > 1$  and  $e^\gamma \leq e^{-2} < 1/2$ . This establishes uniqueness of the root  $\hat{\alpha}$  satisfying (37).

Let us now turn our attention to the bounds in (38). As far as the lower bound is concerned, note that, since  $-2Be^{\frac{\gamma}{\alpha}} < 0$  in the denominator of (37),  $\hat{\alpha} > -\gamma(2 - \gamma)^{-1} = \gamma(\gamma - 2)^{-1}$ , and, as  $\gamma \leq -2$ ,  $\gamma(\gamma - 2)^{-1} \geq 1/2$ . The upper bound requires some more work. First note that  $\hat{\alpha} = -\gamma A / [-\gamma A + 2(A - Be^{\frac{\gamma}{\alpha}})] < 1$ , since  $-\gamma A > 0$  and  $2(A - Be^{\frac{\gamma}{\alpha}}) > 0$ . Hence, let us assume  $0 < \alpha \leq 1$ . As  $-2Be^{\frac{\gamma}{\alpha}}$  is decreasing for  $0 < \alpha \leq 1$ , a first upper bound for  $\hat{\alpha}$  is given by

$$-\frac{\gamma A}{(2 - \gamma)A - 2Be^\gamma}, \quad (42)$$

which is independent of  $\alpha$ . In order to let it be independent of parameters  $\beta$  and  $\delta$  as well, we expand constants  $A$  and  $B$  as in (28) and (29) respectively and differentiate with respect to  $\beta$ :

$$\frac{\partial}{\partial \beta} \left\{ \frac{-\gamma [1 - (1 - \delta)\beta]}{(2 - \gamma) [1 - (1 - \delta)\beta] - 2\delta\beta e^\gamma} \right\} = \frac{-2\gamma\delta e^\gamma}{\{(2 - \gamma) [1 - (1 - \delta)\beta] - 2\delta\beta e^\gamma\}^2},$$

which is clearly positive. Hence, an upper bound of (42), and thus of  $\hat{\alpha}$  as well, is given by (42) itself evaluated at  $\beta = 1$ :

$$-\frac{\gamma A}{(2 - \gamma)A - 2Be^\gamma} \leq -\frac{\gamma}{2 - \gamma - 2e^\gamma},$$

which is the upper bound in (38), and the proof is complete. ■

**Lemma 3** Under Assumption 1,  $g'''(\alpha) \geq 0$  for  $\gamma(\gamma - 2)^{-1} \leq \alpha \leq 1$ , which, by condition (38) of Lemma 2, implies that  $g'''(\alpha) \geq 0$  for  $\hat{\alpha} \leq \alpha \leq 1$ .

**Proof.** By using the notation (31),  $g'''$  can be written as follows:

$$g'''(\alpha) = -3h''(\alpha) + (1 - \alpha)h'''(\alpha),$$

which, since

$$h''(\alpha) = -\frac{h'(\alpha)}{\alpha^2} [h'(\alpha)\alpha^2 + 2\alpha + \gamma], \quad (43)$$

$$h'''(\alpha) = -\frac{h''(\alpha)}{\alpha^2} [2h'(\alpha)\alpha^2 + 2\alpha + \gamma] + 2\frac{\alpha + \gamma}{\alpha^3}h'(\alpha), \quad (44)$$

can be expanded to

$$g'''(\alpha) = -\left\{3 + \frac{1 - \alpha}{\alpha^2} [2h'(\alpha)\alpha^2 + 2\alpha + \gamma]\right\} h''(\alpha) + 2\frac{(1 - \alpha)(\alpha + \gamma)}{\alpha^3} h'(\alpha),$$

where (44) has been used. To study the inequality  $g'''(\alpha) \geq 0$ , we multiply the last expression by  $\alpha^2$  and substitute  $h''(\alpha)$  as in (43) to get:

$$\{3\alpha^2 + (1 - \alpha) [2h'(\alpha)\alpha^2 + 2\alpha + \gamma]\} \frac{h'(\alpha)}{\alpha^2} [h'(\alpha)\alpha^2 + 2\alpha + \gamma] + 2\frac{1 - \alpha}{\alpha} (\alpha + \gamma) h'(\alpha) \geq 0,$$

which, multiplying both sides by  $\alpha^2/h'(\alpha)$  and recalling that  $h'(\alpha) < 0$ , reduces to

$$\{3\alpha^2 + (1 - \alpha) [2h'(\alpha)\alpha^2 + 2\alpha + \gamma]\} [h'(\alpha)\alpha^2 + 2\alpha + \gamma] + 2\alpha(1 - \alpha)(\alpha + \gamma) \leq 0. \quad (45)$$

In order to solve (45), first note that, since  $\gamma \leq -2$  and  $h' < 0$ , for  $\alpha \leq 1$  the second factor in the first term of the LHS,  $h'(\alpha)\alpha^2 + 2\alpha + \gamma$ , and the second term in the sum,  $2\alpha(1 - \alpha)(\alpha + \gamma)$ , are negative and nonpositive respectively. Therefore, we only need to establish for what values of  $0 < \alpha \leq 1$  the first factor in the first term of the sum,  $3\alpha^2 + (1 - \alpha) [2h'(\alpha)\alpha^2 + 2\alpha + \gamma]$ , turns out to be nonnegative; i.e., after substituting  $h'(\alpha)$  as in (33) and some rearrangements, we must solve the following inequality:

$$\alpha^2 + 2\alpha + (1 - \alpha)\gamma \frac{A + Be^{\frac{\gamma}{\alpha}}}{A - Be^{\frac{\gamma}{\alpha}}} \geq 0,$$

which, since  $A - Be^{\frac{\gamma}{\alpha}} > 0$ , is equivalent to

$$(\alpha^2 + 2\alpha) \left(A - Be^{\frac{\gamma}{\alpha}}\right) + (1 - \alpha)\gamma \left(A + Be^{\frac{\gamma}{\alpha}}\right) \geq 0. \quad (46)$$

We now show that the LHS of (46) is increasing in  $\alpha$ . A direct computation of the derivative with respect to  $\alpha$  of the LHS leads to the following inequality:

$$(2\alpha + 2) \left(A - Be^{\frac{\gamma}{\alpha}}\right) - \gamma \left(A + Be^{\frac{\gamma}{\alpha}}\right) + \frac{2 + \gamma}{\alpha} \gamma Be^{\frac{\gamma}{\alpha}} + \left(1 - \frac{\gamma}{\alpha^2}\right) \gamma Be^{\frac{\gamma}{\alpha}} \geq 0, \quad (47)$$

where all terms on the LHS are positive but the last one. Since

$$\frac{\partial}{\partial \alpha} \left[\left(1 - \frac{\gamma}{\alpha^2}\right) \gamma Be^{\frac{\gamma}{\alpha}}\right] = (-\alpha^2 + 2\alpha + \gamma) \frac{\gamma^2 Be^{\frac{\gamma}{\alpha}}}{\alpha^4} < 0,$$

as  $\gamma^2 Be^{\frac{\gamma}{\alpha}}/\alpha^4 > 0$  while  $-\alpha^2 + 2\alpha + \gamma < 0$  for  $\gamma \leq -2$ , a lower bound for the LHS in (47) is given by:

$$\begin{aligned}
& (2\alpha + 2) \left( A - Be^{\frac{\gamma}{\alpha}} \right) - \gamma \left( A + Be^{\frac{\gamma}{\alpha}} \right) + \frac{2 + \gamma}{\alpha} \gamma Be^{\frac{\gamma}{\alpha}} + (1 - \gamma) \gamma Be^{\gamma} \\
& \geq (2\alpha + 2) \left( A - Be^{\frac{\gamma}{\alpha}} \right) - \gamma \left( A + Be^{\frac{\gamma}{\alpha}} \right) + (1 - \gamma) \gamma Be^{\gamma} \\
& = 2\alpha \left( A - Be^{\frac{\gamma}{\alpha}} \right) + A + (1 - \gamma) A - (2 + \gamma) Be^{\gamma} + (1 - \gamma) \gamma Be^{\gamma} \\
& \geq 2\alpha \left( A - Be^{\frac{\gamma}{\alpha}} \right) + A + (1 - \gamma) (A + \gamma Be^{\gamma}), \tag{48}
\end{aligned}$$

where in the second line we dropped the third term of the first line (being it nonnegative) and in the fourth line we dropped  $-(2 + \gamma) Be^{\gamma} \geq 0$ . The first two terms in (48) are clearly positive; thus, as  $1 - \gamma > 0$ , we want to establish that  $A + \gamma Be^{\gamma} > 0$ , or, equivalently, that  $-\gamma e^{\gamma} < A/B$ . Since, by (30),  $A/B > 1$ , a sufficient condition is  $-\gamma e^{\gamma} < 1$ ; as  $(\partial/\partial\gamma)(-\gamma e^{\gamma}) = -(1 + \gamma) e^{\gamma} > 0$  for  $\gamma \leq -2$ , it is enough that  $2e^{-2} < 1$ , which is true. Hence, we have just established that the LHS of (46) is increasing in  $\alpha$ .

Therefore, in order to inequality (46) to hold true for  $\gamma(\gamma - 2)^{-1} \leq \alpha \leq 1$ , it is sufficient that it holds in  $\alpha = \gamma(\gamma - 2)^{-1}$ . By substituting  $\alpha = \gamma(\gamma - 2)^{-1}$  in (46) and rearranging terms we get:

$$\frac{3\gamma - 4}{\gamma - 2} (A - Be^{\gamma-2}) - 2(A + Be^{\gamma-2}) \geq 0,$$

which is convenient to rewrite as

$$\frac{3\gamma - 4}{\gamma - 2} \geq 2 \frac{A + Be^{\gamma-2}}{A - Be^{\gamma-2}}. \tag{49}$$

As the LHS is strictly decreasing in  $\gamma$ , we can let  $\gamma = -2$  into the LHS and get the following sufficient condition for (49):

$$\frac{A + Be^{\gamma-2}}{A - Be^{\gamma-2}} \leq \frac{5}{4}. \tag{50}$$

In order to find an upper bound for the LHS in (50), we substitute the constants  $A$  and  $B$  as in (28) and (29) respectively and differentiate with respect to  $\beta$ ; after rearranging terms we get:

$$\frac{\partial}{\partial\beta} \left[ \frac{1 - (1 - \delta)\beta + \delta\beta e^{\gamma-2}}{1 - (1 - \delta)\beta - \delta\beta e^{\gamma-2}} \right] = \frac{2\delta e^{\gamma-2}}{[1 - (1 - \delta)\beta - \delta\beta e^{\gamma-2}]^2},$$

which is clearly positive. Thus, we can set  $\beta = 1$  in the LHS of (50) so to get

$$\frac{1 + e^{\gamma-2}}{1 - e^{\gamma-2}} \leq \frac{5}{4},$$

which boils down to  $\gamma \leq 2 - \ln 9 \simeq -0.197$ , which clearly holds under Assumption 1.

This is enough to establish inequality (46), which, in turn, is sufficient for inequality (45) to hold; therefore,  $g'''(\alpha) \geq 0$  for  $\gamma(\gamma - 2)^{-1} \leq \alpha \leq 1$  and the proof is complete. ■

**Corollary 1** *Under Assumption 1, the function  $\phi(\alpha)$  defined in (24) is such that  $\phi'''(\alpha) > 0$  for  $\hat{\alpha} \leq \alpha \leq 1$ .*

**Proof.** It is immediately seen that the third derivative of  $\phi$  can be written as follows:

$$\phi'''(\alpha) = e^{g(\alpha)} \left\{ g'''(\alpha) + 3g''(\alpha)g'(\alpha) + [g'(\alpha)]^3 \right\},$$

which is positive for  $\hat{\alpha} \leq \alpha \leq 1$  as, by Lemma 3,  $g'''(\alpha) \geq 0$  and, by Lemmas 1 and 2, also  $g'(\alpha) > 0$  (and thus  $[g'(\alpha)]^3 > 0$ ) and  $g''(\alpha) \geq 0$ . ■

Now we turn our attention to the function  $\psi$  defined in (24).



**Lemma 4** Under Assumption 1, the following holds for the function  $\psi(\alpha) = -pe^{\gamma\frac{1-\alpha}{\alpha}}$  defined in (24):

- i)  $\psi'(\alpha) < 0$  for  $\alpha > 0$ ;
- ii)  $\psi''(\alpha) < 0$  for  $0 < \alpha < -\gamma/2$ ,  $\psi''(\alpha) > 0$  for  $\alpha > -\gamma/2$  and  $\psi''(-\gamma/2) = 0$ ;
- iii)  $\psi'''(\alpha) < 0$  for  $0 < \alpha < \tilde{\alpha}$ ,  $\psi'''(\alpha) > 0$  for  $\tilde{\alpha} < \alpha \leq -\gamma/2$  and  $\psi'''(\tilde{\alpha}) = 0$ , where

$$\tilde{\alpha} = -\frac{3 - \sqrt{3}}{6}\gamma. \quad (51)$$

Note that, under Assumption 1,  $6/(\sqrt{3} - 3) \leq \gamma \leq -2$  implies that<sup>15</sup>

$$0 < 1 - 1/\sqrt{3} \leq \tilde{\alpha} \leq 1. \quad (52)$$

Thus,  $\psi$  is strictly decreasing, it is concave for  $0 < \alpha < -\gamma/2$  and convex for  $\alpha > -\gamma/2$ , while its second derivative is decreasing for  $0 < \alpha < \tilde{\alpha}$  and increasing for  $\tilde{\alpha} \leq \alpha \leq -\gamma/2$ .

**Proof.** Direct computation yields:

$$\psi'(\alpha) = \frac{p\gamma}{\alpha^2} e^{\gamma\frac{1-\alpha}{\alpha}}, \quad (53)$$

$$\psi''(\alpha) = -\frac{p\gamma}{\alpha^4} e^{\gamma\frac{1-\alpha}{\alpha}} (2\alpha + \gamma), \quad (54)$$

$$\psi'''(\alpha) = \frac{p\gamma}{\alpha^6} e^{\gamma\frac{1-\alpha}{\alpha}} (6\alpha^2 + 6\gamma\alpha + \gamma^2).$$

Condition (i) holds as  $\gamma < 0$ . Since  $-(p\gamma/\alpha^4) e^{\gamma\frac{1-\alpha}{\alpha}} > 0$ , the sign of  $\psi''$  is entirely determined by the sign of  $2\alpha + \gamma$  and condition (ii) follows accordingly. Similarly, as  $(p\gamma/\alpha^6) e^{\gamma\frac{1-\alpha}{\alpha}} < 0$ , the sign of  $\psi'''$  is entirely determined by the sign of  $6\alpha^2 + 6\gamma\alpha + \gamma^2$ , which has the two roots  $-\gamma(3 - \sqrt{3})/6$  and  $-\gamma(3 + \sqrt{3})/6$ ; since  $-\gamma/2 < -\gamma(3 + \sqrt{3})/6$ , condition (iii) is established. ■

## 6.2 Case (i) of Proposition 1: $\gamma = -2$

In this section we start to assemble all the information gathered in the last section in order to prove the first part of Proposition 1. First of all, notice that when  $\gamma = -2$ ,  $-\gamma/2 = 1$ , and thus Lemma 4 (ii) establishes that  $\psi$  is strictly convex for  $\alpha \geq 1$ ; coupled with Lemma 2, which implies that  $\phi$  is strictly convex for  $\alpha \geq 1$  as well, this is enough to guarantee that the function  $f(\alpha) = \phi(\alpha) + \psi(\alpha) - (1 - p)$  defined in (23) is strictly convex on  $[1, +\infty)$ . As  $f(1) = 0$ , this means that the solution of the second inequality of system (26) is a nontrivial interval if and only if  $f'(1) < 0$ . Note that, as  $\lim_{\alpha \rightarrow +\infty} f(\alpha) = +\infty$ , any such interval is always closed and has 1 as its left endpoint.

More problematic is the analysis over  $(0, 1]$ , that is, the study of the first inequality of system (26). Recalling that, from (38) in Lemma 2,  $1/2 < \hat{\alpha} < 1$ , the idea underlying the proof is to partition  $(0, 1]$  into two subintervals,  $(0, \hat{\alpha}]$  and  $[\hat{\alpha}, 1]$ , and then exploit the monotonicity and curvature properties which are specific for the functions  $\phi$  and  $\psi$  on such subintervals, as established by the lemmas in the previous section. We start with two lemmas which are specific for the scenario in which  $\gamma = -2$ .

**Lemma 5** When  $\gamma = -2$ , if both Assumption 1 and condition (20) in Proposition 1 (i) hold, then the function  $g$  defined in (27) has the property that  $g'(\alpha) < 1$  for  $0 < \alpha \leq \hat{\alpha}$ .

<sup>15</sup>Note that  $1 - 1/\sqrt{3} \simeq 0.4226$ .

**Proof.** By Lemma 2,  $g'$  is decreasing on  $(0, \hat{\alpha}]$ , thus  $g'(\alpha) < \lim_{\alpha \rightarrow 0^+} g'(\alpha) = -\ln[1 - (1 - \delta)\beta]$  for  $0 < \alpha \leq \hat{\alpha}$ . Hence we must show that  $-\ln[1 - (1 - \delta)\beta] \leq 1 \iff \beta \leq (1 - e^{-1}) / (1 - \delta)$  holds true. Since, by condition (20),  $\beta \leq [1 - \exp(-\sqrt{3.9781p})] / (1 - \delta)$ , it is sufficient to show that

$$\frac{1 - \exp(-\sqrt{3.9781p})}{1 - \delta} \leq \frac{1 - e^{-1}}{1 - \delta} \iff p \leq \frac{1}{3.9781}.$$

A slightly stronger condition is  $p \leq 1/4$ , which certainly holds under condition (18) of Assumption 1. As a matter of fact, by substituting  $\gamma = -2$  into (19) – which is the expanded version of (18) – and recalling that  $t < 1$ , we find that

$$p \leq \frac{0.08634t}{1 - 0.13534t} \leq \frac{1}{4} < \frac{1}{3.9781},$$

and the proof is complete. ■

**Lemma 6** *Under the same assumptions of Lemma 5, if  $0 < \alpha_0 \leq \hat{\alpha}$  is a stationary point for the function  $f$  defined in (23),  $f'(\alpha_0) = \phi'(\alpha_0) + \psi'(\alpha_0) = 0$ , then a smooth function  $f_U : (0, \alpha_0] \rightarrow \mathbb{R}$  exists such that  $f_U(\alpha_0) = f(\alpha_0)$ ,  $f'_U(\alpha_0) = f'(\alpha_0)$ ,  $f_U(\alpha) > f(\alpha)$  for  $0 < \alpha < \alpha_0$  and  $f'_U(\alpha) > 0$  for  $0 < \alpha < \alpha_0$ . In other words, an upper bound  $f_U$  of  $f$  exists on  $(0, \alpha_0]$  such that  $f_U$  is strictly larger than  $f$  and it is strictly increasing on  $(0, \alpha_0)$ .*

**Proof.** Choose  $0 < \alpha_0 \leq \hat{\alpha}$  such that  $f'(\alpha_0) = \phi'(\alpha_0) + \psi'(\alpha_0) = 0$ . Define the first order Taylor approximation of the function  $g$  defined in (27) at  $\alpha = \alpha_0$ :

$$T_0(\alpha) = g(\alpha_0) + g'(\alpha_0)(\alpha - \alpha_0).$$

By Lemma 2,  $g$  is strictly concave on  $(0, \hat{\alpha}]$ , thus  $T_0$  is a strict upper bound of  $g$  on  $(0, \hat{\alpha})$ , and therefore, also the function defined by

$$\phi_U(\alpha) = e^{T_0(\alpha)} = e^{g(\alpha_0) + g'(\alpha_0)(\alpha - \alpha_0)} \quad (55)$$

is such that  $\phi_U \geq \phi$  on  $(0, \hat{\alpha}]$ , with strict inequality on  $(0, \hat{\alpha})$ . Our goal is to show that the upper bound of function  $f$  defined on  $(0, \hat{\alpha}]$  as

$$f_U(\alpha) = \phi_U(\alpha) + \psi(\alpha) + (1 - p),$$

is strictly increasing on  $(0, \alpha_0)$ , that is,

$$f'_U(\alpha) = \phi'_U(\alpha) + \psi'(\alpha) > 0 \quad (56)$$

for  $0 < \alpha < \alpha_0$ .

From Lemma 4 (iii) we know that  $\psi'$  is strictly concave on  $(0, \tilde{\alpha}]$ , where  $\tilde{\alpha}$  is given by (51), as  $\psi''' < 0$  for  $0 < \alpha < \tilde{\alpha}$ , while it is strictly convex for  $\tilde{\alpha} < \alpha \leq -\gamma/2 = 1$ . When  $\gamma = -2$ , however,  $\tilde{\alpha} = 1 - 1/\sqrt{3} < 1/2 < \hat{\alpha}$  [see (38)]; therefore, we need to study separately the two cases  $0 < \alpha_0 \leq 1 - 1/\sqrt{3}$  and  $1 - 1/\sqrt{3} < \alpha_0 \leq \hat{\alpha}$ .

1. Assume that  $0 < \alpha_0 \leq 1 - 1/\sqrt{3} = \tilde{\alpha}$ . Then  $\psi'$  is strictly concave on  $(0, \alpha_0]$ ;  $\phi'_U$ , however, it is not, being it an exponential function, as can be seen by differentiating  $\phi_U$  in (55):  $\phi'_U(\alpha) = g'(\alpha_0) e^{T_0(\alpha)}$ . Therefore, we shall linearize it by taking its first order Taylor approximation at  $\alpha = \alpha_0$ :

$$T_L(\alpha) = \phi'_U(\alpha_0) + \phi''_U(\alpha_0)(\alpha - \alpha_0) = g'(\alpha_0) e^{g(\alpha_0)} [1 + g'(\alpha_0)(\alpha - \alpha_0)].$$

Clearly, being  $\phi'_U$  a strictly convex function,  $T_L$  is a lower bound for  $\phi'_U$ ; specifically:  $T_L(\alpha_0) = \phi'_U(\alpha_0)$  and  $T_L(\alpha) < \phi'_U(\alpha)$  for  $0 < \alpha < \alpha_0$ . As  $T_L$  is also linear, the function defined by

$$L(\alpha) = T_L(\alpha) + \psi'(\alpha), \quad (57)$$

turns out to be a strictly concave lower bound of  $f'_U$ , defined in (56), on  $(0, \alpha_0]$ :  $L(\alpha) \leq f'_U(\alpha)$  for  $0 < \alpha \leq \alpha_0$ . Hence, in order to establish that  $f'_U > 0$  on  $(0, \alpha_0)$ , it is sufficient to show that  $L > 0$  on  $(0, \alpha_0)$ . By strict concavity of  $L$ , a sufficient condition is that  $\min\{\lim_{\alpha \rightarrow 0^+} L(\alpha), L(\alpha_0)\} \geq 0$ . But, on one hand, as, by (53),  $\lim_{\alpha \rightarrow 0^+} \psi'(\alpha) = \lim_{\alpha \rightarrow 0^+} (p\gamma/\alpha^2) e^{\gamma \frac{1-\alpha}{\alpha}} = 0$ ,

$$\lim_{\alpha \rightarrow 0^+} L(\alpha) = \lim_{\alpha \rightarrow 0^+} [T_L(\alpha) + \psi'(\alpha)] = g'(\alpha_0) e^{g(\alpha_0)} [1 - g'(\alpha_0) \alpha_0]$$

is strictly positive since  $0 < \alpha_0 < 1$  and, by Lemmas 1 and 5,  $0 < g'(\alpha_0) < 1$ . On the other hand, by construction,  $L(\alpha_0) = \phi'(\alpha_0) + \psi'(\alpha_0)$ , which equals zero by assumption. Hence inequality (56) is established for  $0 < \alpha_0 \leq 1 - 1/\sqrt{3}$ .

2. Now assume that  $\tilde{\alpha} = 1 - 1/\sqrt{3} < \alpha_0 \leq \hat{\alpha}$  and consider again the function  $L$  defined in (57), which, while being strictly concave on  $(0, \tilde{\alpha}]$ , it turns out to be strictly convex on  $(\tilde{\alpha}, \alpha_0]$  as, by Lemma 4 (iii),  $\psi'$  is. Our strategy is to extend the argument of the previous point to the whole interval  $(0, \alpha_0]$  by showing that, under condition (20) in Proposition 1 (i),  $L$  must be strictly decreasing over  $(\tilde{\alpha}, \alpha_0]$ ; therefore, provided that  $L'(\alpha) < 0$  for  $\tilde{\alpha} < \alpha \leq \alpha_0$ , once again the condition  $\min\{\lim_{\alpha \rightarrow 0^+} L(\alpha), L(\alpha_0)\} \geq 0$  guarantees that inequality (56) holds also when  $\alpha_0 > \tilde{\alpha}$ . Hence, let us study the sign of the derivative of the lower bound  $L$ :

$$L'(\alpha) = T'_L(\alpha) + \psi''(\alpha) = [g'(\alpha_0)]^2 e^{g(\alpha_0)} + 4 \frac{p}{\alpha^4} e^{-2 \frac{1-\alpha}{\alpha}} (\alpha - 1), \quad (58)$$

where  $\psi''(\alpha)$  have been expanded as in (54) computed for  $\gamma = -2$ . We must thus establish that the RHS in (58) is strictly negative.

As  $\alpha_0 \leq \hat{\alpha} < 1$ , by definition (27)  $g(\alpha_0) < 0$ , and thus  $e^{g(\alpha_0)} < 1$ , yielding  $[g'(\alpha_0)]^2$  as a first upper bound of  $[g'(\alpha_0)]^2 e^{g(\alpha_0)}$ ; moreover, since by Lemmas 1 and 2  $g'$  is positive and decreasing on  $(0, \hat{\alpha}]$ , an upper bound of  $g'(\alpha_0)$  is  $\lim_{\alpha \rightarrow 0^+} g'(\alpha) = -\ln[1 - (1 - \delta)\beta]$ . Thus, an upper bound for the first term in the RHS of (58) is  $\{\ln[1 - (1 - \delta)\beta]\}^2$ .

As far as the second term in the RHS of (58) is concerned, from Lemma 4 (iii) we know that  $\psi''$  is increasing for  $\alpha > \tilde{\alpha}$ ; therefore a useful upper bound for  $\psi''$  is given by the number  $\psi''(\hat{\alpha})$ , which, however, cannot be computed directly. Thus we shall instead employ the upper bound of  $\hat{\alpha}$  provided in (38) of Lemma 2 for  $\gamma = -2$ :  $\gamma/(\gamma - 2 + 2e^\gamma) = (2 - e^{-2})^{-1}$ . By substituting  $(2 - e^{-2})^{-1}$  in the argument of  $\psi''(\alpha)$  in the RHS of (58), after some algebra we get:

$$\psi''\left(\frac{1}{2 - e^{-2}}\right) = \frac{(1 - e^{-2})(2e^{-2} - 4)^3 e^{-2(1 - e^{-2})}}{2} p \simeq -3.9781p.$$

Using the two upper bounds just found, we get the following inequality which holds on  $(\tilde{\alpha}, \alpha_0]$ :

$$L'(\alpha) = T'_L(\alpha) + \psi''(\alpha) < \{\ln[1 - (1 - \delta)\beta]\}^2 - 3.9781p;$$

hence, a sufficient condition for  $L' < 0$  on  $(\tilde{\alpha}, \alpha_0]$  is  $\{\ln[1 - (1 - \delta)\beta]\}^2 < 3.9781p$ , which is condition (20) in Proposition 1 (i). This is enough for  $\min\{\lim_{\alpha \rightarrow 0^+} L(\alpha), L(\alpha_0)\} \geq 0$  to hold true also when  $1 - 1/\sqrt{3} < \alpha_0 \leq \hat{\alpha}$ .

We have thus shown that inequality (56) holds for  $0 < \alpha < \alpha_0$ , where  $\alpha_0$  can be any stationary point of  $f$  in  $(0, \hat{\alpha}]$ , and the proof is complete. ■

**Corollary 2** *Under the assumptions of Lemma 6,  $f$  can have at most one stationary point  $\alpha_0$  in  $(0, \hat{\alpha}]$  which must be a maximum, while there cannot be any stationary point which is a minimum for  $f$  in  $(0, \hat{\alpha}]$ . Either  $f'(\alpha) > 0$  for  $0 < \alpha \leq \hat{\alpha}$ , or (a unique)  $0 < \alpha_0 < \hat{\alpha}$  exists such that  $f'(\alpha_0) = 0$ ; in the latter case  $f$  turns out to be strictly increasing on  $(0, \alpha_0)$  and strictly decreasing on  $(\alpha_0, \hat{\alpha}]$ .*

**Proof.** First of all, note that  $\lim_{\alpha \rightarrow 0^+} f'(\alpha) = -[1 - (1 - \delta)\beta] \ln[1 - (1 - \delta)\beta] > 0$ ; therefore, if  $f$  has no stationary points on  $(0, \hat{\alpha}]$  it must be increasing over there. Now let us suppose that a stationary point exists and argue by contradiction: let  $0 < \alpha_0 \leq \hat{\alpha}$  be such that  $f'(\alpha_0) = 0$  which is a minimum point of  $f$ . But, by Lemma 6, an upper bound  $f_U$  of  $f$  exists on  $(0, \alpha_0]$  such that  $f_U(\alpha_0) = f(\alpha_0)$  and  $f_U(\alpha) > f(\alpha)$  for  $0 < \alpha < \alpha_0$  which is strictly increasing on  $(0, \alpha_0)$ ; therefore,  $f$  itself must be strictly increasing at least on a (left) neighborhood of  $\alpha_0$ , which contradicts our assumption. The same argument rules out existence of multiple maxima on  $(0, \alpha_0]$  as well, since this would imply the existence of stationary points which are minima among the maximum points. ■

**Proof of Proposition 1 (i).** Corollary 2 states that the function  $f$  defined in (23) is either increasing or has at most one maximum point on  $(0, \hat{\alpha}]$ . Since, for  $\gamma = -2$ ,  $\tilde{\alpha} = 1 - 1/\sqrt{3} < 1/2 < \hat{\alpha} < 1 = -\gamma/2$ , Corollary 1 and Lemma 4 (iii) establish that both  $\phi$  and  $\psi$  have positive third derivative on  $[\hat{\alpha}, 1]$ , and thus  $f''' > 0$  on  $[\hat{\alpha}, 1]$  accordingly. Finally, Lemmas 2 and 4 (ii) establish that both  $\phi$  and  $\psi$  are strictly convex for  $\alpha \geq 1$ , which implies that  $f$  is strictly convex on  $[1, +\infty)$  as well. By combining these three properties we deduce that the function  $f$  can be either increasing on all  $(0, +\infty)$ , or it can have at most one maximum point  $\alpha_0$  such that  $0 < \alpha_0 < 1$  and one minimum point  $\alpha_1$  such that  $\alpha_1 > \alpha_0$ . More specifically,  $\alpha_0$  can either satisfy  $0 < \alpha_0 \leq \hat{\alpha}$  or  $\hat{\alpha} < \alpha_0 < 1$ , where the number  $\hat{\alpha}$  is defined in (37) of Lemma 2 and, by calculating the bounds in (38) for  $\gamma = -2$ , is such that  $0.5 < \hat{\alpha} < 0.5363$ ; while  $\alpha_1$  can either be such that  $\alpha_0 < \alpha_1 \leq 1$  or  $\alpha_1 > 1$ , depending on whether  $f'(1) \geq 0$  or  $f'(1) < 0$  respectively. Recalling also that  $f(1) = 0$  and  $\lim_{\alpha \rightarrow +\infty} f(\alpha) = +\infty$ , the following possible scenarios can occur, all defining the solution set of system (26) as either the empty set or an interval.

1.  $f$  is strictly increasing on  $(0, +\infty)$  and thus it crosses the abscissa on  $\alpha = 1$  from below, i.e.,  $f'(1) > 0$ ; in this case system (26) has an empty solution set.
2. There is one maximum point  $0 < \alpha_0 < 1$  and one minimum point  $\alpha_1$  for  $f$  such that  $\alpha_0 < \alpha_1 < 1$  and  $f(\alpha_0) < 0$ ;  $f$  crosses the abscissa on  $\alpha = 1$  from below, i.e.,  $f'(1) > 0$ , and system (26) has an empty solution set.
3. There is one maximum point  $0 < \alpha_0 < 1$  and one minimum point  $\alpha_1$  for  $f$  such that  $\alpha_0 < \alpha_1 < 1$  and  $f(\alpha_0) = 0$ ;  $f$  crosses the abscissa on  $\alpha = 1$  from below, i.e.,  $f'(1) > 0$ , and the solution set of (26) is the singleton  $\{\alpha_0\}$ .
4. There is one maximum point  $0 < \alpha_0 < 1$  and one minimum point  $\alpha_1$  for  $f$  such that  $\alpha_0 < \alpha_1 < 1$  and  $f(\alpha_0) > 0$ ; since  $\alpha_1 < 1$ , once again  $f$  crosses the abscissa on  $\alpha = 1$  from below, i.e.,  $f'(1) > 0$ , and the solution set of (26) can be either
  - (a) the closed interval  $[\alpha_\ell, \alpha_r]$ , with  $0 < \alpha_\ell < \alpha_0 < \alpha_r < \alpha_1 < 1$ , or
  - (b) the left-open interval  $(0, \alpha_r]$ , with  $0 < \alpha_0 < \alpha_r < \alpha_1 < 1$ .
5. There is one maximum point  $0 < \alpha_0 < 1$  and the minimum point for  $f$  is  $\alpha_1 = 1$ ; then  $f'(1) = 0$ , and the solution set of (26) can be either

- (a) the closed interval  $[\alpha_\ell, 1]$ , with  $\alpha_\ell > 0$ , or
  - (b) the left-open interval  $(0, 1]$ .
6. There is one maximum point  $0 < \alpha_0 < 1$  and one minimum point  $\alpha_1$  for  $f$  such that  $\alpha_1 > 1$ ; then  $f$  crosses the abscissa on  $\alpha = 1$  from above, *i.e.*,  $f'(1) < 0$ , and the solution set of (26) can be either
- (a) the closed interval  $[\alpha_\ell, \alpha_r]$ , with  $0 < \alpha_\ell < \alpha_0 < 1 < \alpha_1 < \alpha_r$ , or
  - (b) the left-open interval  $(0, \alpha_r]$ , with  $0 < \alpha_0 < 1 < \alpha_1 < \alpha_r$ .

Since the six cases discussed include all possibilities, the proof is complete. ■

Figure 3 illustrates the last proof by showing some solution sets  $I$  of system (26) when  $\gamma = -2$ . All figures has been plotted using the values of parameters employed in the second example of Section 4:  $\gamma = -2$ ,  $t = 0.44$  and  $p = 0.05$ ; while parameter  $\beta$  takes decreasing values from figure 3(a) to Figure 3(f). Figure 3(a) corresponds to case 1 in the proof, while figure 3(b) matches case 2: in both cases the solution set is empty,  $I = \emptyset$ . Figure 3(c) shows case 3, in which  $I$  is the singleton  $\{\alpha_0\}$ . Figure 3(d) explains case 4a, while figures 3(e) and 3(f) illustrate case 5a, in which  $\alpha_1 = \alpha_r = 1$ , and case 6a, in which  $\alpha_r > 1$  (and thus  $f'(1) < 0$ ), respectively: all the last three cases produce a closed interval as solution set,  $I = [\alpha_\ell, \alpha_r]$ , with left endpoint  $\alpha_\ell$  strictly larger than zero. Finally, figure 3(f) corresponds to case 6b, when the solution set is an interval which is left-open:  $I = (0, \alpha_r]$ .

### 6.3 Case (ii) of Proposition 1: $6/(\sqrt{3}-3) \leq \gamma < -2$

Since  $\gamma < -2$  implies  $-\gamma/2 > 1$ , by Lemma 4 (ii) the function  $\psi$  defined in (25) turns out to be concave on  $[1, -\gamma/2]$ ; however, Lemma 2 states that the function  $\phi$  defined in (24) is convex on  $[1, -\gamma/2]$ . Since no information is available on the sign of  $\phi'''$  on the interval  $[1, -\gamma/2]$  (note that Corollary 1 holds only for  $\hat{\alpha} \leq \alpha \leq 1$ ), if  $\gamma < -2$  in principle nothing can be said on the behavior of  $f$  defined in (23) on the interval  $[1, -\gamma/2]$ . The task of condition (21) in Proposition 1 (ii) is to overcome this impasse by letting  $f$  to be, if not convex, at least quasiconvex on  $[1, +\infty)$ , so that its lower contour sets are still intervals. Therefore, under condition (21), the analysis on  $[1, +\infty)$  remains the same as in the previous section. However, as we have seen in Section 4, there is a price to pay, as condition (21) turns out to be much more restrictive than condition (20).

**Lemma 7** *Under Assumption 1, if condition (21) in Proposition 1 (ii) holds the function  $f$  defined in (23) is quasiconvex on  $[1, +\infty)$ ; specifically, the set  $\{\alpha \geq 1 : f(\alpha) \leq 0\}$  is always a nonempty (nontrivial) closed interval.*

**Proof.** Consider the following linear upper bound  $\chi$  of function  $\phi$  on  $[1, -\gamma/2]$ :

$$\chi(\alpha) = 1 + \frac{\exp\left\{\left(1 + \frac{\gamma}{2}\right) \ln[1 - (1 - \delta + e^{-2}\delta)\beta]\right\} - 1}{-\left(1 + \frac{\gamma}{2}\right)} (\alpha - 1). \quad (59)$$

As  $\phi(1) = 1$  and  $\phi(-\gamma/2) = \exp\{(1 + \gamma/2) \ln[1 - (1 - \delta + e^{-2}\delta)\beta]\}$ ,  $\chi$  is the expression of the line defined by the two points  $(1, \phi(1))$  and  $(-\gamma/2, \phi(-\gamma/2))$ . Since  $\phi$  is strictly convex on  $[1, +\infty)$ ,  $\chi$  is strictly larger than  $\phi$  on  $(1, -\gamma/2)$ .

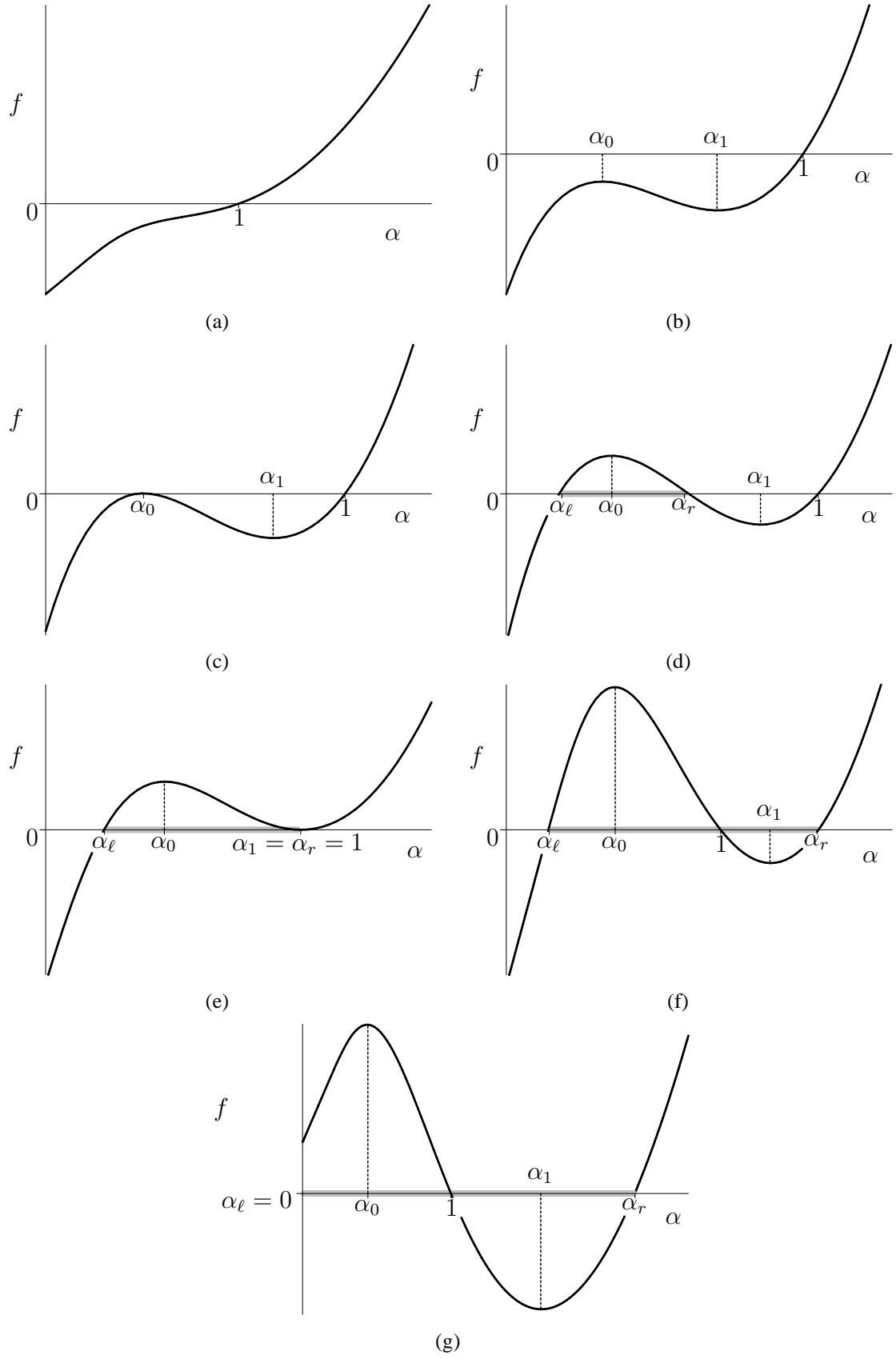


FIGURE 3: the solution set  $I$  of system (26) is either empty or a (possibly nontrivial) interval; in (a) and (b)  $I = \emptyset$ , in (c)  $I = \{\alpha_0\}$ , in (d), (e) and (f)  $I = [\alpha_\ell, \alpha_r]$ , while in (g)  $I = (0, \alpha_r]$ .

Define the upper bound  $\bar{f}$  of  $f$  as follows:

$$\bar{f}(\alpha) = \begin{cases} f(\alpha) & \text{if } 0 < \alpha < 1 \text{ and } \alpha > -\gamma/2 \\ \chi(\alpha) + \psi(\alpha) - (1-p) & \text{if } 1 \leq \alpha \leq -\gamma/2, \end{cases} \quad (60)$$

where  $\chi(\alpha)$  is defined in (59). The function  $\bar{f}$  defined in (60) is such that  $\bar{f}(1) = f(1)$ ,  $\bar{f}(-\gamma/2) = f(-\gamma/2)$ ,  $\bar{f}(\alpha) > f(\alpha)$  for  $1 < \alpha < -\gamma/2$ , and is strictly concave on  $[1, -\gamma/2]$ . Therefore, by assuming that

$$\chi'(1) + \psi'(1) \leq 0, \quad (61)$$

$\bar{f}$  turns out to be strictly decreasing on  $[1, -\gamma/2]$ , which, since  $\bar{f}(1) = f(1) = 0$ , in turns implies that  $f(\alpha) < 0$  on  $(1, -\gamma/2]$ . Since, by Lemmas 2 and 4 (ii),  $f$  is strictly convex on  $(-\gamma/2, +\infty)$ , condition (61) is thus sufficient to establish that  $f$  is quasiconvex on  $[1, +\infty)$ . Using (53) to evaluate  $\psi'(1) = p\gamma$  and after some algebra, (61) boils down to condition (21), and the proof is complete. ■

When  $\gamma < -2$ , however, something changes also on the left of  $\alpha = 1$ . As a matter of fact, the two constants  $\hat{\alpha}$  and  $\tilde{\alpha}$  introduced in Lemmas 2 and 4 (iii) both increase as  $\gamma$  decreases, with  $\hat{\alpha}$  increasing faster than  $\tilde{\alpha}$ , as the next lemma explains in detail. This reshuffles the arguments used in the previous section also on the interval  $(0, 1]$ , which thus need to be reassessed.

**Lemma 8** *Under Assumption 1, a unique value  $\gamma^* < -2$  exists such that  $\tilde{\alpha} = \hat{\alpha}$  when  $\gamma = \gamma^*$ ,  $\tilde{\alpha} < \hat{\alpha}$  if  $\gamma^* < \gamma < -2$  and  $\tilde{\alpha} > \hat{\alpha}$  if  $\gamma < \gamma^*$ , where  $\tilde{\alpha}$  and  $\hat{\alpha}$  are defined in (51) and in (37) respectively. Bounds independent of the parameters are given by:*

$$-2.7497 \simeq -2 \left[ \frac{1}{\sqrt{3}-1} + \exp\left(-\frac{6}{3-\sqrt{3}}\right) \right] < \gamma^* < -\frac{2}{\sqrt{3}-1} \simeq -2.7321. \quad (62)$$

**Proof.** By substituting  $\hat{\alpha}$  with  $\tilde{\alpha}$ , defined in (51), into the expression (37) and rearranging terms we consider the following function of  $\gamma$ :

$$\eta(\gamma) = \gamma + \frac{2}{\sqrt{3}-1} + 2e^{-\frac{6}{3-\sqrt{3}}} \frac{B}{A} = \gamma + \frac{2}{\sqrt{3}-1} + \frac{2p\beta e^{-\frac{6}{3-\sqrt{3}}}}{(1-\beta)t[p + (1-p)e^\gamma] + p\beta}, \quad (63)$$

where in the last equality the constants  $A$  and  $B$  have been expanded as in (28) and (29) respectively and the definition of  $\delta$  in (15) has been used. Direct computation yields:

$$\eta'(\gamma) = 1 - \frac{2p\beta e^{-\frac{6}{3-\sqrt{3}}} (1-\beta)t(1-p)e^\gamma}{\{(1-\beta)t[p + (1-p)e^\gamma] + p\beta\}^2},$$

which is clearly positive under Assumption 1. Thus, the function  $\eta$  defined in (63) is strictly increasing in  $\gamma$  and it can have at most one root  $\gamma^*$  in  $(-\infty, -2]$ , which solves  $\eta(\gamma) = \gamma + 2/(\sqrt{3}-1) + 2e^{-\frac{6}{3-\sqrt{3}}}(B/A) = 0$ . As  $0 < B/A < 1$ , the bounds in (62) follow immediately. ■

Note that, since  $6/(\sqrt{3}-3) \simeq -4.7321$ , by (62) it is immediately seen that  $\gamma^* > 6/(\sqrt{3}-3)$  must hold.

By Lemma 8,  $\tilde{\alpha} \leq \hat{\alpha}$  if  $\gamma^* \leq \gamma < -2$ . This means that the argument developed through Lemmas 5 and 6 and in Corollary 2 in Section 6.2, and thus in the proof of Proposition 1 (i), carries over also when  $\gamma^* \leq \gamma < -2$ ; we only need to adjust Lemmas 5 and 6 in order let them hold also in this case. The next Lemma actually generalizes Lemma 5 for all  $6/(\sqrt{3}-3) \leq \gamma < -2$  under condition (21) in Proposition 1 (ii), while the following one, Lemma 10, is specific for the case  $\gamma^* \leq \gamma < -2$ .

**Lemma 9** When  $6/(\sqrt{3}-3) \leq \gamma < -2$ , if both Assumption 1 and condition (21) in Proposition 1 (ii) hold, then the function  $g$  defined in (27) has the property that  $g'(\alpha) < 1$  for  $0 < \alpha \leq \hat{\alpha}$ , where  $\hat{\alpha}$  is defined in (37).

**Proof.** Recalling the proof of Lemma 5, we must show that  $\beta \leq (1 - e^{-1}) / (1 - \delta)$  holds true also when  $\gamma^* \leq \gamma < -2$ . Under condition (21), it is sufficient to show that

$$\frac{1 - \exp \left\{ \frac{2 \ln [1 + (1 + \gamma/2) p \gamma]}{2 + \gamma} \right\}}{1 - (1 - e^{-2}) \delta} \leq \frac{1 - e^{-1}}{1 - \delta},$$

or, equivalently, that

$$\frac{1 - \delta}{1 - (1 - e^{-2}) \delta} \left\{ 1 - \exp \left[ \frac{2 \ln (1 + p \gamma + p \gamma^2 / 2)}{2 + \gamma} \right] \right\} \leq 1 - e^{-1}. \quad (64)$$

As  $(1 - \delta) / [1 - (1 - e^{-2}) \delta] < 1$ , a sufficient condition for (64) is

$$\exp \left[ \frac{2 \ln (1 + p \gamma + p \gamma^2 / 2)}{2 + \gamma} \right] \geq e^{-1} \quad \Longleftrightarrow \quad p \leq \frac{2 \left( e^{-\frac{2+\gamma}{2}} - 1 \right)}{(2 + \gamma) \gamma}. \quad (65)$$

Condition (18) of Assumption 1, or, more precisely, condition (19), provides a useful upper bound for  $p$ ; since an upper bound for the RHS in (19) is obtained by letting  $t = 1$  in its expression, we obtain the following:

$$p \leq \frac{\left[ 1 + \frac{\gamma e^\gamma}{(1 - e^\gamma)^2} \right] e^\gamma t}{1 - \left[ 1 + \frac{\gamma e^\gamma}{(1 - e^\gamma)^2} \right] (1 - e^\gamma) t} < \frac{\left[ 1 + \frac{\gamma e^\gamma}{(1 - e^\gamma)^2} \right] e^\gamma}{1 - \left[ 1 + \frac{\gamma e^\gamma}{(1 - e^\gamma)^2} \right] (1 - e^\gamma)} = \frac{(1 - e^\gamma)^2 + \gamma e^\gamma}{(1 - e^\gamma) (1 - e^\gamma - \gamma)}. \quad (66)$$

Combining (65) and (66), it is easily seen that

$$\frac{2 \left( e^{-\frac{2+\gamma}{2}} - 1 \right)}{(2 + \gamma) \gamma} > \frac{(1 - e^\gamma)^2 + \gamma e^\gamma}{(1 - e^\gamma) (1 - e^\gamma - \gamma)} \quad (67)$$

holds true for all  $6/(\sqrt{3}-3) \leq \gamma < -2$ , as can be checked by plotting both terms in (67) as functions of the only variable  $\gamma$  with Maple software. As (67) implies both (65) and (64), the proof is complete. ■

### 6.3.1 The argument when $\gamma^* \leq \gamma < -2$

The following Lemma exploits both Lemma 9 and condition (21) in Proposition 1 (ii) in order to extend Lemma 6 to the case  $\gamma^* \leq \gamma < -2$ .

**Lemma 10** Let  $\gamma^* \leq \gamma < -2$ . Under the same assumptions of Lemma 9, if  $0 < \alpha_0 \leq \hat{\alpha}$  is a stationary point for the function  $f$  defined in (23),  $f'(\alpha_0) = \phi'(\alpha_0) + \psi'(\alpha_0) = 0$ , then a smooth function  $f_U : (0, \alpha_0] \rightarrow \mathbb{R}$  exists such that  $f_U(\alpha_0) = f(\alpha_0)$ ,  $f'_U(\alpha_0) = f'(\alpha_0)$ ,  $f_U(\alpha) > f(\alpha)$  for  $0 < \alpha < \alpha_0$  and  $f'_U(\alpha) > 0$  for  $0 < \alpha < \alpha_0$ . In other words, an upper bound  $f_U$  of  $f$  exists on  $(0, \alpha_0]$  such that  $f_U$  is strictly larger than  $f$  and it is strictly increasing on  $(0, \alpha_0)$ .



**Proof.** Since  $\tilde{\alpha} < \hat{\alpha}$  if  $\gamma^* < \gamma < -2$ , the first part of the proof, *i.e.*, the case in which  $0 < \alpha_0 \leq \tilde{\alpha}$ , is exactly the same as in the proof of Lemma 6, one has only to replace  $1 - 1/\sqrt{3}$  with  $\tilde{\alpha} = -(3 - \sqrt{3})\gamma/6$  and use Lemma 9 instead of Lemma 5. Note that the same argument works also in the special case  $\gamma = \gamma^*$ , in which  $\tilde{\alpha} = \hat{\alpha}$ .

As far as the second part is concerned, *i.e.*, when  $\gamma^* < \gamma < -2$  and  $\tilde{\alpha} < \alpha_0 \leq \hat{\alpha}$ , we must show that under condition (21) in Proposition 1 (ii) the function  $L$  defined in (57) is strictly decreasing on  $(\tilde{\alpha}, \alpha_0]$ . To this purpose, let us study the sign of the derivative of  $L$  as in (58) for  $\gamma^* < \gamma < -2$ :

$$L'(\alpha) = T_L'(\alpha) + \psi''(\alpha) = [g'(\alpha_0)]^2 e^{g(\alpha_0)} - \frac{p\gamma}{\alpha^4} e^{\gamma \frac{1-\alpha}{\alpha}} (2\alpha + \gamma), \quad (68)$$

where  $\psi''(\alpha)$  have been expanded as in (54). We must establish that the RHS in (68) is strictly negative. We shall consider the same upper bounds for both the first and the second term in the RHS of (58):  $\{\ln[1 - (1 - \delta)\beta]\}^2$  and  $\psi''[\gamma/(\gamma - 2 + 2e^\gamma)]$  respectively, where the argument of  $\psi''$  is the upper bound of  $\hat{\alpha}$  provided in (38) of Lemma 2. After some algebra we get:

$$\psi''\left(\frac{\gamma}{\gamma - 2 + 2e^\gamma}\right) = -\frac{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}}{\gamma^2}.$$

Therefore, a sufficient condition for having the RHS in (68) strictly negative is

$$\{\ln[1 - (1 - \delta)\beta]\}^2 - \frac{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}}{\gamma^2} \leq 0,$$

which can be rearranged as follows:

$$\beta \leq \frac{1 - \exp\left[-\sqrt{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}/\gamma^2}\right]}{1 - \delta}.$$

Under condition (21), it is thus sufficient to show that

$$\frac{1 - \exp\left[\frac{2 \ln(1 + p\gamma + p\gamma^2/2)}{2 + \gamma}\right]}{1 - (1 - e^{-2})\delta} \leq \frac{1 - \exp\left[-\sqrt{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}/\gamma^2}\right]}{1 - \delta},$$

or, equivalently, that

$$\frac{1 - \delta}{1 - (1 - e^{-2})\delta} \left\{ 1 - \exp\left[\frac{2 \ln(1 + p\gamma + p\gamma^2/2)}{2 + \gamma}\right] \right\} \leq 1 - \exp\left[-\sqrt{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}/\gamma^2}\right].$$

Since  $(1 - \delta) / [1 - (1 - e^{-2})\delta] < 1$ , a sufficient condition is

$$\exp\left[\frac{2 \ln(1 + p\gamma + p\gamma^2/2)}{2 + \gamma}\right] \geq \exp\left[-\sqrt{p(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}/\gamma^2}\right],$$

which, after some algebra, is equivalent to

$$\theta(p, \gamma) = p - \frac{\left[\frac{2\gamma \ln(1 + p\gamma + p\gamma^2/2)}{2 + \gamma}\right]^2}{(\gamma + 2e^\gamma)(\gamma - 2 + 2e^\gamma)^3 e^{-2(1-e^\gamma)}} \geq 0.$$

Note that the LHS of the last inequality depends only on the two parameters  $p$  and  $\gamma$ ; hence we can label it as a function of two variables,  $\theta(p, \gamma)$ . Since its expression is too tough to handle analytically we rely on graphic inspection by means of Maple software, which confirms that  $\theta(p, \gamma) \geq 0$  for all  $0 < p < 1$  and  $\gamma < -2$ . This completes the proof as it is sufficient for condition (68) to hold true. ■

**Proof of Proposition 1 (ii) for  $\gamma^* \leq \gamma < -2$ .** With Lemmas 9 and 10 replacing Lemmas 5 and 6 in Section 6.2, Corollary 2 still applies, and the proof remains identical to that for case (i) of Proposition 1 on  $(0, 1]$ . Moreover, Lemma 7 extends the argument in the proof of Proposition 1 (i) also on the interval  $[1, +\infty)$  by establishing that, if condition (21) holds true, the set  $\{\alpha \geq 1 : f(\alpha) \leq 0\}$  is always a nonempty (nontrivial) closed interval. Note that under condition (21) the solution set of system (26) cannot be empty. ■

### 6.3.2 The argument when $6/(\sqrt{3}-3) \leq \gamma < \gamma^*$

Clearly,  $\gamma < \gamma^*$  implies  $-\gamma/2 > 1$ ; thus Lemma 7 still applies and condition (21) in Proposition 1 (ii) guarantees that the function  $f$  defined in (23) is quasiconvex on  $[1, +\infty)$ , so that its lower contour sets are intervals. However, when  $\gamma < \gamma^*$  Lemma 8 states that the constants  $\hat{\alpha}$  and  $\tilde{\alpha}$  defined in (37) of Lemma 2 and in (51) of Lemma 4 (iii) respectively, are such that  $\hat{\alpha} < \tilde{\alpha}$ . Moreover, as  $\gamma$  becomes smaller, the constant  $\tilde{\alpha}$  becomes larger, until it reaches the value  $\tilde{\alpha} = 1$ , which, by definition (51), corresponds to the value  $6/(\sqrt{3}-3)$  for parameter  $\gamma$ . Hence, unlike the case  $\gamma^* \leq \gamma \leq -2$ , now the interval  $(0, 1]$  must be partitioned into the following three intervals:

$$(0, 1] = (0, \hat{\alpha}] \cup [\hat{\alpha}, \tilde{\alpha}] \cup [\tilde{\alpha}, 1],$$

with  $\hat{\alpha} < \tilde{\alpha} \leq 1$ .

While on  $(0, \hat{\alpha}]$  and, when  $\tilde{\alpha} < 1$ , on  $[\tilde{\alpha}, 1]$  the arguments discussed in the previous sections still apply, on  $[\hat{\alpha}, \tilde{\alpha}]$  not only, as shown in Lemmas 2 and 4 (ii), the functions  $\phi$  and  $\psi$  defined in (24) and (25) are respectively convex and concave, but also their third derivatives have opposite sign, as prescribed by Corollary 1 and 4 (iii). Therefore, the argument in the proof of case (i) of Proposition 1 does not apply anymore on  $[\hat{\alpha}, \tilde{\alpha}]$ . We shall follow a new strategy in order to fill this gap: the next lemma will establish that, if condition (21) holds, the function  $f$  turns out to be concave on  $[\hat{\alpha}, \tilde{\alpha}]$ .

**Lemma 11** *Let  $6/(\sqrt{3}-3) \leq \gamma < \gamma^*$ . Under Assumption 1, if condition (21) in Proposition 1 (ii) holds, then the function  $f$  defined in (23) is strictly concave for  $\hat{\alpha} \leq \alpha \leq \tilde{\alpha}$ , where  $\hat{\alpha}$  and  $\tilde{\alpha}$  are defined in (37) and (51) respectively, and are such that  $\hat{\alpha} < \tilde{\alpha} \leq 1$ .*

**Proof.** Since, by Corollary 1,  $\phi''' > 0$  for  $\hat{\alpha} \leq \alpha \leq 1$ ,  $\phi''(\alpha) \leq \phi''(1)$  holds true whenever  $\hat{\alpha} \leq \alpha \leq 1$ . Also, since, by Lemma 4 (iii),  $\psi'''(\alpha) \leq 0$  for  $0 < \alpha \leq \tilde{\alpha}$ ,  $\psi''(\alpha) \leq \psi''(\hat{\alpha}) < \psi''[\gamma/(\gamma-2)]$  whenever  $\hat{\alpha} \leq \alpha \leq \tilde{\alpha}$ , where the last inequality uses the lower bound for  $\hat{\alpha}$  in (38). Hence, a sufficient condition for  $f''(\alpha) = \phi''(\alpha) + \psi''(\alpha) < 0$  on  $[\hat{\alpha}, \tilde{\alpha}]$  is the following:

$$\phi''(1) + \psi''\left(\frac{\gamma}{\gamma-2}\right) < 0,$$

which can be expanded as

$$-\frac{2\gamma\delta\beta e^\gamma}{1-(1-\delta)\beta-\delta\beta e^\gamma} + \{\ln[1-(1-\delta)\beta-\delta\beta e^\gamma]\}^2 < \frac{(\gamma-2)^3 p e^{-2}}{\gamma}, \quad (70)$$

where the LHS has been obtained by substituting  $\alpha = 1$  in the expression of  $\phi''$ ,

$$\phi''(\alpha) = e^{g(\alpha)} \left\{ g''(\alpha) + [g'(\alpha)]^2 \right\},$$

with  $g$ ,  $g'$  and  $g''$  defined in (27), (32) and (39) respectively, and the RHS has been obtained by substituting  $\alpha = \gamma/(\gamma - 2)$  in (54).

Let us search for some useful upper bound for the LHS in (70). First note that for  $\gamma < -2$  the following inequalities hold:

$$\frac{1}{1 - (1 - \delta)\beta - \delta\beta e^\gamma} < \frac{1}{1 - (1 - \delta)\beta - \delta\beta e^{-2}},$$

$$\{\ln[1 - (1 - \delta)\beta - \delta\beta e^\gamma]\}^2 < \{\ln[1 - (1 - \delta)\beta - \delta\beta e^{-2}]\}^2,$$

which provide a first upper bound. Next, note that the RHS in both inequalities above are strictly increasing in  $\beta$ , therefore we can discard parameter  $\beta$  by taking its maximum value yielded by condition (21), so that:

$$1 - (1 - \delta)\beta - \delta\beta e^{-2} = 1 - [1 - (1 - e^{-2})\delta]\beta = \exp\left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\},$$

and thus an upper bound for the LHS in (70) is given by

$$-\left[\frac{2\gamma\delta e^\gamma}{1 - (1 - e^{-2})\delta}\right] \frac{1 - \exp\left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}}{\exp\left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}} + \left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}^2.$$

Finally, we need to get rid also of parameter  $\delta$ ; to this purpose, since it is immediately seen that  $\delta/[1 - (1 - e^{-2})\delta]$  is increasing in  $\delta$ , condition (18) of Assumption 1 allows us to replace  $\delta$  with  $1 + \gamma e^\gamma / (1 - e^\gamma)^2$ , thus eventually providing our final upper bound for the LHS in (70):

$$-\left\{\frac{2\gamma e^\gamma [(1 - e^\gamma)^2 + \gamma e^\gamma]}{[(1 - e^\gamma)^2 + \gamma e^\gamma] e^{-2} - \gamma e^\gamma}\right\} \frac{1 - \exp\left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}}{\exp\left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}} + \left\{\frac{2\ln[1 + (1 + \gamma/2)p\gamma]}{2 + \gamma}\right\}^2.$$

Since the RHS in (70) is increasing in  $p$ , there is no hope to discard parameter  $p$  from the whole inequality; as a matter of fact, we shall rely once again on graphic inspection on the relevant ranges for parameters  $\gamma$  and  $p$ . Recall that an upper bound for  $p$  is given by condition (19) of Assumption 1 computed in  $t = 1$  [see also inequality (66) in the proof of Lemma 9]:

$$p < \frac{(1 - e^\gamma)^2 + \gamma e^\gamma}{(1 - e^\gamma)(1 - e^\gamma - \gamma)} = \bar{p}(\gamma). \quad (71)$$

To conclude the proof, the plot by Maple software of the difference between the LHS and the RHS of (70) as a function of the two variables  $\gamma$  and  $p$  shows that it is clearly positive for all the relevant values, that is, on the ranges<sup>16</sup>  $6/(\sqrt{3} - 3) \leq \gamma \leq 2/(1 - \sqrt{3})$  and  $0 < p \leq \bar{p}(\gamma)$ , with  $\bar{p}(\gamma)$  defined in (71). ■

**Proof of Proposition 1 (ii) for  $6/(\sqrt{3} - 3) \leq \gamma < \gamma^*$ .** Note that Lemma 6 holds also when  $6/(\sqrt{3} - 3) \leq \gamma < \gamma^*$ , as the first part of its proof is sufficient to cover the whole interval  $(0, \hat{\alpha}]$  when  $\hat{\alpha} < \tilde{\alpha}$ ; the only tool required is Lemma 9, which, as a matter of fact, does apply for all  $6/(\sqrt{3} - 3) \leq \gamma < -2$ . Thus, Corollary 2 still applies and states that  $f$  can have, if any, at most one maximum point in  $(0, \hat{\alpha}]$ . Lemma 11 establishes that the function  $f$  is strictly concave on  $[\hat{\alpha}, \tilde{\alpha}]$ , where  $\tilde{\alpha} \leq 1$ . As long as  $\tilde{\alpha} < 1$ , Corollary 1 and Lemma 4 (iii) together state that both  $\phi$  and  $\psi$  have

<sup>16</sup>Recall that, by Lemma 8,  $\gamma^* < 2/(1 - \sqrt{3})$ .

positive third derivative on  $[\tilde{\alpha}, 1]$ , and thus  $f''' > 0$  on  $[\tilde{\alpha}, 1]$  accordingly.<sup>17</sup> Finally, Lemma 7 covers the interval  $[1, +\infty)$  by establishing that, if condition (21) holds true, the set  $\{\alpha \geq 1 : f(\alpha) \leq 0\}$  is always a nonempty (nontrivial) closed interval.

Therefore, if 1)  $f$  can have at most one maximum point in  $(0, \hat{\alpha}]$ , 2)  $f$  is strictly concave on  $[\hat{\alpha}, \tilde{\alpha}]$ , 3)  $f''' > 0$  on  $[\tilde{\alpha}, 1]$  whenever  $\tilde{\alpha} < 1$  and 4) the set  $\{\alpha \geq 1 : f(\alpha) \leq 0\}$  is a closed interval, then the behavior of  $f$  is that described in the proof of Proposition 1 (i) also when  $6/(\sqrt{3} - 3) \leq \gamma < \gamma^*$ , and the proof is complete. Recall that under condition (21) the solution set of system (26) is always nonempty. ■

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<sup>17</sup>Note that when  $\tilde{\alpha} = 1$  [i.e., when  $\gamma = 6/(\sqrt{3} - 3)$ ], there is no need of this argument as, by Lemma 11,  $f$  turns out to be concave on all  $[\hat{\alpha}, 1]$ .